Kac's Program for the Landau Equation

Xuanrui Feng

Beijing International Center for Mathematical Research
Peking University

October 25, 2025 at CSIAM TM 33

Description of the Physical World

Basic Question: How can we describe a complicated system of a vast number of identical interacting particles, such as a rarefied gas or a plasma?

Microscopic scale: state vector $(x_1, v_1, \cdots, x_N, v_N) \in (\mathbb{R}^3 \times \mathbb{R}^3)^N$ governed by Newton's second law of motion. Complete and detailed, but impossible to track and analyze mathematically.

Macroscopic scale: spatial observables such as the mass $\rho(t, x)$, the velocity $u(t, x)$ and the temperature $T(t, x)$ governed by fluid equations. Losing information about the velocity distribution.

Mesoscopic scale: one-particle density function $f(t, x, v)$ governed by kinetic equations. Intermediate between the above two scales.

Foundings of Kinetic Theory

Maxwell (1867): started the study of dynamics of gases; wrote down the first version of the Boltzmann equation and discovered the Maxwellian equilibrium.

Boltzmann (1872): wrote down the explicit formula of the Boltzmann equation; established the famous H-theorem and explained the convergence to the Maxwellian equilibrium.

Landau (1936): wrote down the Landau equation as an alternative to the Boltzmann equation when the charged particles perform Coulomb collisions in a plasma, by assuming that the grazing collisions prevail.

Hilbert (1900): asked for a rigorous derivation from the microscopic Newton's law to the mesoscopic kinetic equations and then to the macroscopic fluid equations in his sixth problem at the ICM.

The essentially difficult first part remains open for the Landau equation.
Bobylev-Pulvirenti-Saffirio 2013 CMP: a consistency result.
Lanford 1975, Deng-Hani-Ma 2024, 2025: from hard-sphere to Boltzmann.

Kac's Program

Kac (1956): introduced his seminal program to derive *spatially homogeneous* kinetic equations from many-particle systems.

Kac's model: starting from a conservative stochastic model, which simplifies the dynamics to a random Markovian process but also preserves some key features.

Boltzmann collisional operator: $B(|z|, \cos\theta) = \Gamma(|z|) \cdot b(\cos\theta)$.
Initial state space: N i.i.d. velocities $V_1, \cdots, V_N$ with initial law f_0 in $\mathbb{R}^3$.
Collision rate: for each pair of indices (i, j), choose a random collision time T_{ij} according to the exponential distribution with parameter $\Gamma(|V_i - V_j|)$. Let $T = \min T_{ij}$ be the collision time and change the corresponding pair of velocities.
Post-collisional state space:

$$V_i' = \frac{V_i + V_j}{2} + \frac{|V_i - V_j|}{2}\sigma, \quad V_j' = \frac{V_i + V_j}{2} - \frac{|V_i - V_j|}{2}\sigma,$$

where $\sigma \in \mathbb{S}^2$ is drawn according to the law $b(\cos\theta_{ij})$. θ_{ij} is the deviation angle between the pre-collisional and post-collisional velocities.

Kac's Program for the Landau Equation

The Landau equation is formally obtained by passing the Boltzmann equation to the *grazing limit*.
We can also pass to the *grazing limit* for the Boltzmann master equation of Kac's stochastic model to obtain the Landau master equation.

$$
\begin{tikzcd}
\text{Boltzmann master equation} \arrow[r, "\text{grazing limit}"] \arrow[d, "\text{mean-field limit}"'] & \text{Landau master equation} \arrow[d, "\text{mean-field limit (goal)}"] \\
\text{Boltzmann equation} \arrow[r, "\text{grazing limit}"'] & \text{Landau equation}
\end{tikzcd}
$$

Landau Equation and Master Equation

Landau Equation: integral-differential form

$$\partial_t f = \nabla_v \cdot \int_{\mathbb{R}^3} a(v-w) \cdot (\nabla_v - \nabla_w) f(v) f(w) \, \mathrm{d}w,$$

or more commonly used divergence form

$$\partial_t f = \nabla \cdot [(a * f)\nabla f - (b * f) f].$$

Here $a(z) = |z|^\gamma (|z|^2 \mathrm{Id} - z \otimes z)$ and $b(z) = \nabla \cdot a(z) = -2|z|^\gamma z$.
The parameter $\gamma \in [-3, 1]$ represents the regularity of the interaction forces.
$\gamma = -3$ corresponds to the most important Coulomb interaction case.

Landau Master Equation:

$$\partial_t F_N = \frac{1}{N} \sum_{i<j}^{N} (\nabla_{v_i} - \nabla_{v_j}) \cdot \Big(a(v_i - v_j) \cdot (\nabla_{v_i} - \nabla_{v_j}) F_N \Big).$$

We take $\gamma = -3$ in this talk.

Kac's Program for the Landau Equation

Theorem (F.-Z.Wang (2025))

Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$ be a probability density. Let $F_N \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^{3N}))$ be the unique bounded weak solution to the Landau master equation with initial data $f_0^{\otimes N}$. Let $f \in C^1((0,\infty); \mathcal{S}(\mathbb{R}^3)) \cap L^\infty([0,\infty); L^1 \cap L^\infty(\mathbb{R}^3))$ be the unique bounded smooth solution to the Landau equation with initial data f_0.
Assume that f_0 has finite weighted Fisher information and high-order L^1 moment:

$$\int_{\mathbb{R}^3} \langle v \rangle^3 |\nabla \log f_0|^2 f_0 \, \mathrm{d}v < \infty, \quad \int |v|^m f_0 \, \mathrm{d}v < \infty \text{ for some } m > 6.$$

Then for any $T > 0$, we have propagation of chaos: for any $k \geq 1$, the k-marginal $F_{N,k}$ converges weakly to $f^{\otimes k}$ as $N \to \infty$ on $[0,T] \times \mathbb{R}^{3k}$:

$$\int_{\mathbb{R}^{3k}} (F_{N,k} - f^{\otimes k}) \varphi^{\otimes k} \, \mathrm{d}v_1 \cdots \mathrm{d}v_k \to 0$$

as $N \to \infty$, for any $\varphi \in C_c^\infty(\mathbb{R}^3)$ and any $t \in [0,T]$. Here $\langle v \rangle = \sqrt{1 + |v|^2}$.

Further Discussion

1. Quantitative propagation of chaos for the Landau master equation; Convergence in better sense.
2. Similar results for the Boltzmann equation or the Vlasov-Poisson equation?
3. Central limit theorem/ Gaussian fluctuation/ Large deviation principles for kinetic equations.
4. Derivation of spatially inhomogeneous kinetic equations/ from Newton dynamics under low-density limit or weak-coupling limit.

References

S. Mischler and C. Mouhot, Kac's program in kinetic theory. **Invent. Math. 193** (2013) 1-147.

D. Bresch, M. Duerinckx and P.-E. Jabin, A duality method for mean-field limits with singular interactions. arXiv preprint arXiv:2402.04695 (2024).

X. Feng and Z. Wang, Kac's program for the Landau equation. arXiv preprint arXiv:2506.14309 (2025).

Thank you!