(一). 选择题

14. C. 15. D 原式 $= 2\int_0^{\pi/2} \sin x\, dx = 2$

16. C 原式 $= \int_0^1 1-x\, dx + \int_1^3 x-1\, dx = \frac{1}{2}+2=\frac{5}{2}$

17. D 原式 $= \frac{1}{2}\int_a^x f'(2t)\, d(2t) = \frac{1}{2}\int_{2a}^{2x} f'(y)\, dy$

18. D 19. A $f(x)=2x^3$. 代入

20. A 21. C 原式 $= \int_a^x f(t+a)\, d(t+a)$

22. D 洛必达. 23. D 24. B

(二). 解答题

11. $\frac{dy}{dx} = \sin x$, $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

12. 由 $y=f(x) \Rightarrow dy = f'(x)\,dx \Rightarrow dy = \sqrt{1+x^2}\, dx$

13. $y = f(x^2) = \int_1^{x^2} \frac{1}{1+t}\, dt \Rightarrow \frac{dy}{dx} = \frac{2x}{1+x^2}$

14. 同理 $\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^4}}$

15. $\Phi'(x) = \frac{\sin^2 x^2}{1+\cos^2 x^2} \cdot 2x$

16 (1) $\int_1^3 x^3\, dx = \frac{1}{4}x^4\Big|_1^3 = 20$

(2) $\int_1^4 \sqrt{x}\, dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_1^4 = \frac{14}{3}$

(3) $\int_\pi^{2\pi} \sin x\, dx = -\cos x\Big|_\pi^{2\pi} = -2$

(4) $\int_0^1 \frac{1}{4t^2-9}\, dt = \frac{1}{12}\ln\left|\frac{2t-3}{2t+3}\right|\Big|_0^1 = -\frac{1}{12}\ln 5$

(5) $\int_{-1}^0 e^{-x}\, dx = -e^{-x}\Big|_{-1}^0 = e-1$

(6) $\int_{-1}^{-2} \frac{x}{x+3}\, dx = x-3\ln|x+3|\Big|_{-1}^{-2} = 3\ln 2-1$

17. $\int_0^{\frac{3}{2}} f(x)\,dx = \int_0^1 x^2 dx + \int_1^{\frac{3}{2}} e^{-x}\, dx = \frac{1}{3} + e^{-1} - e^{-\frac{3}{2}}$

18 (1) $x\in[0,1]$ 时 $x^2 \geq x^3$ 则 $\int_0^1 x^2 dx \geq \int_0^1 x^3 dx$, 进一步 $\int_0^1 x^2 dx > \int_0^1 x^3 dx$

(2) $x \in [1,2]$ 时 $x^3 \geqslant x^2 \Rightarrow \int_1^2 x^3 dx \geqslant \int_1^2 x^2 dx$

(3) $x \in [1,2]$ 时 $\ln x \geqslant \ln^2 x \Rightarrow \int_1^2 \ln x\, dx \geqslant \int_1^2 \ln^2 x\, dx$

(4) $\int_0^1 f(x)\,dx = \int_{-1}^0 g(x)\,dx$. 但 $\int_{-1}^0 f(x)\,dx < 0 < \int_0^1 g(x)\,dx \Rightarrow \int_{-1}^1 f(x)dx < \int_{-1}^1 g(x)dx$

19. $\int_1^4 y\,dx = \int_1^4 2x^2+3x+3\,dx = 73.5$ 均值为 $\frac{73.5}{3} = 24.5$

20. 当 $x \in [10,20]$ 时 $\frac{1}{2x^2} < \frac{x^2}{x^4+x+1} < \frac{1}{x^2}$ 而 $\int_{10}^{20} \frac{1}{x^2} = \frac{1}{20}$. 得证

21 (1) $\int_0^1 x^2\sqrt{1-x^2}\,dx \xlongequal{x=\sin t} \int_0^{\pi/2} \sin^2 t \cos^2 t\, dt = \frac{1}{4}\int_0^{\pi/2} \sin^2 2t\, dt = \frac{\pi}{16}$

(2) $\int_1^e \frac{1+\ln x}{x}\,dx = \int_1^e 1+\ln x\, d\ln x = \frac{3}{2}$

(3) $\int_0^1 \frac{dx}{1+e^x} \xlongequal{e^x=t} \int_1^e \frac{1}{1+t}\cdot\frac{1}{t}\,dt = \int_1^e \frac{1}{t} - \frac{1}{t+1}\,dt = \ln\frac{2e}{1+e}$

(4) $\int_0^1 \sqrt{4-x^2}\,dx = \int_0^{\pi/6} 2\cos t\cdot 2\cos t\,dt = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

(5) $\int_{-2}^0 \frac{dx}{x^2+2x+2} = \int_{-2}^0 \frac{dx}{(x+1)^2+1} = \int_{-1}^1 \frac{dt}{t^2+1} = \frac{\pi}{2}$

(6) $\int_0^1 xe^x\,dx = xe^x\big|_0^1 - \int_0^1 e^x\,dx = 1$

(7) $\int_1^e x\ln x\,dx = \cancel{\int_0^1 e^t\cdot t\,dt}\ \int_0^1 e^t\cdot t\cdot e^t dt = \frac{1}{4}(e^2+1)$

(8) $\int_0^1 x\arctan x\,dx = x^2\arctan x\big|_0^1 - \int_0^1 x\left(\arctan x + \frac{x}{1+x^2}\right)dx$

$\Rightarrow \int_0^1 x\arctan x\,dx = \frac{1}{2}\left(x^2\arctan x\big|_0^1 - \int_0^1 \frac{x^2}{1+x^2}\,dx\right) = \frac{1}{4}(\pi-2)$

(9) $\int_0^{e-1} \ln(x+1)\,dx = x\ln(x+1)\big|_0^{e-1} - \int_0^{e-1}\frac{x}{x+1}\,dx = 1$

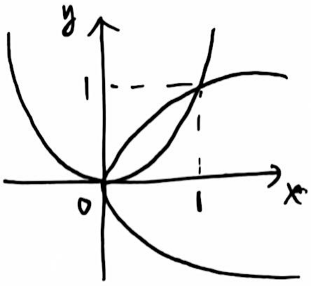
(10) $\int_0^\pi x^3\sin x\,dx = -x^3\cos x\big|_0^\pi + \int_0^\pi 3x^2\cos x\,dx$

$\int_0^\pi 3x^2\cos x\,dx = 3x^2\sin x\big|_0^\pi - \int_0^\pi 6x\sin x\,dx = -6\pi$

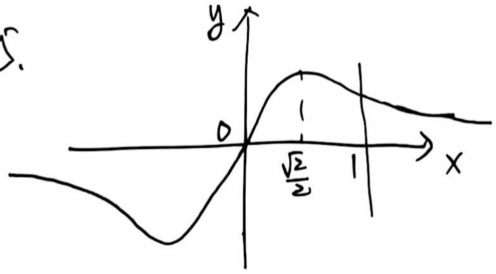
$\int_0^\pi 6x\sin x\,dx = -6x\cos x\big|_0^\pi + \int_0^\pi 6\cos x\,dx = 6\pi$

代入. 原式 $= \pi^3 - 6\pi$

24.



交点为 $(1,1)$. 面积为 $S = \int_0^1 \sqrt{x} - x^2\,dx = \frac{1}{3}$

25. 

$$S=\int_0^1 xe^{-x^2}dx=\frac{1}{2}(1-e^{-1})$$

26. (1) $V=\pi\int_0^1 (x-x^4)\,dx=\frac{3}{10}\pi$

(2) $V=\pi\int_0^{\pi} \sin^2 x\,dx=\frac{\pi^2}{2}$

28. 质量 $M=\int_1^2 e^y\,dy=e^2-e$

29. (1) $\int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}=\int_{-\infty}^{+\infty}\frac{dt}{t^2+1}=\pi$

(2) $\int_e^{+\infty}\frac{1}{x\ln^2 x}dx=\int_e^{+\infty}\frac{1}{\ln^2 x}d\ln x=1$

(3) $\int_{-\infty}^{0}\frac{dx}{(1-2x)^{3/2}}\overset{1-2x=t}{=\!=}\frac{1}{2}\int_1^{+\infty}\frac{1}{t^{3/2}}dt=1$

(4) $\int_1^{+\infty}\frac{dx}{(x+\sqrt{x^2-1})^n}=\int_1^{+\infty}\frac{1}{t^n}\left(\frac{1}{2}-\frac{1}{2t^2}\right)dt=\frac{1}{n^2-1}$

其中. 令 $x+\sqrt{x^2-1}=t\Rightarrow x^2-1=t^2-2tx+x^2\Rightarrow x=\frac{t^2+1}{2t}$

(5) $\int_1^{+\infty}\frac{dx}{x^p}=\begin{cases}\frac{1}{p-1}, & 若\ p>1\\ 发散. & 若\ p\le 1\end{cases}$

(6) $\int_2^{+\infty}\frac{dx}{x^2(1+x)}=\int_2^{+\infty}\frac{1}{x^2}-\frac{1}{x(1+x)}dx=\frac{1}{2}-\ln\frac{3}{2}$