高数D第八次作业解答:

选择题.

10. $\int x\,de^{-x} = xe^{-x} - \int e^{-x}dx = xe^{-x} + e^{-x} + C.$ 选 C.

11. $\int e^{-x} f(e^{-x})\,dx = -\int f(e^{-x})\,de^{-x} = -F(e^{-x}) + C.$ 选 B.

12. $\cancel{\int x f(x) dx = \int f(x)\, d\frac{x^2}{2} = \frac{x^2}{2} f(x) - \int \frac{x^2}{2}\, d f(x)}$

$\cancel{= \frac{x^2}{2} f(x)}$ $\int x f(x) dx = \int x\, d(x\ln x) = x^2\ln x -$

$\int x\ln x\, dx = x^2 \ln x - \int \ln x\, d(\frac{x^2}{2}) = \frac{x^2}{2}\ln x + \int \frac{x}{2} dx$

$= \frac{x^2}{2}\ln x + \frac{x^2}{4} + C.$ 选B.

13. $\int \ln\frac{x}{2} dx = \int \ln x - \ln 2\, dx = x\ln x - x - (\ln 2)x + C = x\ln\frac{x}{2} - x + C.$ 选 C.

解答题.

5. (1) $\int x\sin 2x\, dx = -\frac{1}{2}\int x\, d\cos 2x = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$

(2) $\int x\ln x\, dx = \int \ln x\, d(\frac{x^2}{2}) = \frac{x^2}{2}\ln x - \int \frac{x}{2} dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$

(3) $\int xe^{-x} dx = -\int x\, de^{-x} = -xe^{-x} + \int e^{-x} dx = -(x+1)e^{-x}.$

(4) $\int \ln^2 x\, dx = x\ln^2 x - \int x\, d(\ln^2 x) = x\ln^2 x - 2\int \ln x\, dx = x\ln^2 x - 2x\ln x + 2x + C.$

(5) $\int \arccos x\, dx = x\arccos x + \int \frac{x}{\sqrt{1-x^2}} dx = x\arccos x - \sqrt{1-x^2} + C.$

(6) $\int x\arctan x\,dx = \int \arctan x\,d\left(\frac{x^2}{2}\right) = \frac{x^2}{2}\arctan x - \int \frac{x^2}{2(1+x^2)}\,dx$

$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx = \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C.$

(7) $\int (x^2+2x+1)e^x\,dx = \int (x^2+2x+1)\,de^x = (x^2+2x+1)e^x - \int e^x(2x+2)\,dx$

$= (x^2+2x+1)e^x - 2\int (x+1)\,de^x = (x^2+2x+1)e^x - 2(x+1)e^x + 2e^x + C$

$= (x^2+1)e^x + C.$

(8) $\int (\arcsin x)^2\,dx = x(\arcsin x)^2 - \int \frac{2x}{\sqrt{1-x^2}}\arcsin x\,dx = x(\arcsin x)^2$

$+2\int \arcsin x\,d\sqrt{1-x^2} = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2\int \sqrt{1-x^2}\cdot\frac{dx}{\sqrt{1-x^2}}$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C.$

6\. (1) $\int \frac{x}{\sqrt{x+2}}\,dx = 2\int x\,d\sqrt{x+2} = 2x\sqrt{x+2} - 2\int \sqrt{x+2}\,dx = 2x\sqrt{x+2} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C.$

(2) 令 $t=\tan\frac{x}{2}$，则 $1+\cos x = 2\cos^2\frac{x}{2} = \frac{2}{t^2+1}$. $dx = \frac{2}{t^2+1}\,dt$

$\int \frac{dx}{1+\cos x} = \int \frac{1+t^2}{2}\cdot\frac{2}{t^2+1}\,dt = t + C = \tan\frac{x}{2} + C.$

(3) $\int \sin^4 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{(\cos 2x)^2}{4}\,dx$

$= \frac{x}{4} - \frac{1}{4}\sin 2x + \int \frac{1+\cos 4x}{8}\,dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

(4) $\int \tan^6 x\cdot\sec^4 x\,dx = \int \sec^2 x\,d\left(\frac{1}{7}\tan^7 x\right) = \frac{1}{7}\tan^7 x\cdot\sec^2 x - \frac{2}{7}\int \tan^8 x\cdot\sec^2 x\,dx$

$= \frac{1}{7}\tan^7 x\cdot\sec^2 x - \frac{2}{63}\tan^9 x + C.$

(5) 令 $t = \frac{1}{\sqrt{1+e^x}}$，则 $x = \ln\left(\frac{1}{t^2}-1\right)$，$dx = \frac{t^2}{1-t^2}\cdot\left(-2\frac{1}{t^3}\right)dt = -\frac{2}{t(1-t^2)}\,dt.$

$\int \frac{dx}{\sqrt{1+e^x}} = \int t\cdot\frac{2}{t(t^2-1)}\,dt = \ln\left|\frac{t-1}{t+1}\right| + C = \ln\frac{-1+\sqrt{1+e^x}}{1+\sqrt{1+e^x}} + C.$

(6) 令 $t=\sqrt{1+x^2}$，则 $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+1}+(\sqrt{x^2+1})^3} = \int \frac{1}{t^3+t}\, d\frac{t^2-1}{2} = \int \frac{dt}{t^2+1}$

$= \arctan t + C = \arctan\sqrt{1+x^2} + C.$

(7) $x\geq 0$ 时 $\int e^{|x|}dx = e^x + C$，$x<0$ 时 $\int e^{|x|}dx = -e^{-x}+C$.

$\Rightarrow \int e^{|x|}dx = \operatorname{sgn}(x)\cdot e^{|x|} + C.$

(8) $\displaystyle\int \cos^3 x\,dx = \int (1-\sin^2 x)\, d\sin x = \sin x - \frac{1}{3}(\sin x)^3 + C.$

(9) 令 $x=2\tan t$，$t\in[-\frac{\pi}{2},\frac{\pi}{2}]$，则 $dx = \frac{2}{\cos^2 t}dt$.

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{1}{4\tan^2 t\cdot \frac{2}{\cos t}}\cdot \frac{2}{\cos^2 t}dt = \frac{1}{4}\int \frac{\cos t\, dt}{\sin^2 t}$$

$= -\frac{1}{4\sin t} + C$. $\tan t = \frac{x}{2} \Rightarrow \sin t = \frac{x}{\sqrt{x^2+4}}$. 原式 $= -\frac{\sqrt{x^2+4}}{4x} + C$.

(10) $\displaystyle\int \frac{dx}{\sqrt{x(4-x)}} = 2\int \frac{d\sqrt{x}}{\sqrt{4-x}} \overset{t=\sqrt{x}}{=\!=} 2\int \frac{dt}{\sqrt{4-t^2}} \overset{t=2u}{=\!=} 4\int \frac{du}{2\sqrt{1-u^2}} =$

$2\arcsin u + C = 2\arcsin\frac{\sqrt{x}}{2} + C.$

(11) $\displaystyle\int \frac{x}{\sin^2 x}dx = -\int x\, d\cot x = -x\cot x + \int \cot x\, dx = -x\cot x + \ln|\sin x| + C.$

(12) $\displaystyle\int \frac{x^4}{(1-x^2)^3}dx = \frac{1}{4}\int x^3\, d(1-x^2)^{-2} = \frac{x^3}{4(1-x^2)^2} - \frac{3}{4}\int \frac{x^2}{(1-x^2)^2}dx =$

$\displaystyle\frac{x^3}{4(1-x^2)^2} - \frac{3}{8}\int x\, d(1-x^2)^{-1} = \frac{x^3}{4(1-x^2)^2} - \frac{3x}{8(1-x^2)} + \frac{3}{8}\int \frac{dx}{1-x^2}$

$\displaystyle= \frac{x^3}{4(1-x^2)^2} - \frac{3x}{8(1-x^2)} + \frac{3}{16}\ln\left|\frac{1+x}{1-x}\right| + C.$

7. $f(x) = \frac{1}{x\sqrt{1-x^2}} \Rightarrow \int \frac{dx}{f(x)} = \int x\sqrt{1-x^2}\,dx \overset{t=x^2}{=\!=\!=} \frac{1}{2}\int\sqrt{1-t}\,dt =$

$-\frac{1}{3}(1-t)^{\frac{3}{2}} + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.$

8. $\cancel{\int x^3 f'(x)dx = \int x^3 d\frac{\sin x}{x} = x^2\sin x - 3\int x\sin x\,dx = x^2\sin x + 3x\cos x}$

$\cancel{-3\int\cos x\,dx = x^2\sin x + 3x\cos x - 3\sin x + C}$

$f(x) = \left(\frac{\sin x}{x}\right)' = \frac{\cos x}{x} - \frac{\sin x}{x^2}.$

$\int x^3 f'(x)dx = x^3 f(x) - \int f(x)\cdot 3x^2 dx = x^2\cos x - x\sin x - 3\int(x\cos x - \sin x)\,dx$

$= x^2\cos x - x\sin x - 3\cos x - 3\int x\,d\sin x = x^2\cos x - 4x\sin x - 6\cos x + C.$

9. $y = x^2 + C$ 过 $(2,5) \Rightarrow C = 5 - 2^2 = 1.$ $y = x^2 + 1.$

10. $y = e^{x^2} + \ln|x| + (x^2-1)^3 + C$ 过 $(1, e-2) \Rightarrow C = e - 2 - e = -2.$

$y = e^{x^2} + \ln|x| + (x^2-1)^3 - 2.$