高数D第五次作业解答

1. 设$f(0)=b$. $\forall x>0$，由Lagrange中值定理，$\exists\xi\in(0,x)$，$f'(\xi)=\frac{f(x)-b}{x}$.

由题设$f'(\xi)=a$，故$f(x)=ax+b$. $x<0$时同理可证. □

2. f在$[0,4]$上连续，$f'(x)=\sqrt{4-x}-\frac{x}{2\sqrt{4-x}}$在$(0,4)$上存在，且$f(0)=f(4)=0$，符合条件.

$f'(x_0)=\sqrt{4-x_0}-\frac{x_0}{2\sqrt{4-x_0}}=0 \Rightarrow x_0=\frac{8}{3}$.

3. $f(0)=0$，$f(1)=1$. $\Rightarrow \exists x_0\in(0,1)$，$f'(x_0)\cdot(1-0)=f(1)-f(0)\Rightarrow f'(x_0)=1$.

$f'(x)=3x^2=1 \Rightarrow x_0=\frac{1}{\sqrt{3}}$.

4. $f(x)=x^3+x-1$. $f(0)=-1$，$f(1)=1$. 由$f(0)<0<f(1)$及f在$[0,1]$上连续知$f(x)=0$在$[0,1]$上有根. 又若$\exists\ 0\le x_1<x_2\le 1$，$f(x_1)=f(x_2)=0$，由罗尔定理知$\exists\xi\in(x_1,x_2)$，$f'(\xi)=0$. 但$f'(x)=3x^2+1>0$，矛盾！故存在唯一的根$x\in[0,1]$. □

5. (1) $f(x)=e^x-x-1$，$f(0)=0$. $\forall x>0$，由Lagrange中值定理，$\exists\xi\in(0,x)$，$f'(\xi)=\frac{f(x)}{x}$. 又$f'(\xi)=e^{\xi}-1>0$（因$\xi>0$），故$\frac{f(x)}{x}>0 \Rightarrow f(x)>0$.

$\forall x<0$，$\exists\xi\in(x,0)$，$f'(\xi)=\frac{f(x)}{x}$. 又$f'(\xi)=e^{\xi}-1<0$（因$\xi<0$），故$\frac{f(x)}{x}<0\Rightarrow f(x)>0$.

故总有$e^x>1+x$ $(x\neq 0)$. □

(2) $f(x)=x-\ln(1+x)$，$f(0)=0$. $\forall x>0$，$\exists\xi\in(0,x)$，$f'(\xi)=\frac{f(x)}{x}$. 又$f'(\xi)=1-\frac{1}{1+\xi}>0$（因$\xi>0$），

故$f(x)>0$，即$x>\ln(1+x)$.

$g(x)=\ln(1+x)-\frac{x}{x+1}$，$g(0)=0$. $\forall x>0$，$\exists\xi\in(0,x)$，$g'(\xi)=\frac{g(x)}{x}$. 又

$g'(\xi)=\frac{1}{1+\xi}-\frac{1}{(1+\xi)^2}>0$（因$\xi>0$），故$g(x)>0$，即$\ln(1+x)>\frac{x}{1+x}$. □

6. (1) $f(x)=\arctan x-\arcsin\dfrac{x}{\sqrt{1+x^2}}$.

$$f'(x)=\frac{1}{1+x^2}-\frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}\cdot\frac{\sqrt{1+x^2}-\frac{x^2}{\sqrt{1+x^2}}}{1+x^2}=\frac{1}{1+x^2}-\sqrt{1+x^2}\cdot\frac{\frac{1}{\sqrt{1+x^2}}}{1+x^2}=0.$$

$\Rightarrow$ $f(x)$为恒等于常数的函数. $f(0)=0\Rightarrow f(x)\equiv 0$. □

(2) $f(x)=\arcsin x+\arccos x$.

$f'(x)=\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{\sqrt{1-x^2}}=0$. $\Rightarrow$ $f(x)$恒为常数. $f(0)=\dfrac{\pi}{2}\Rightarrow f(x)\equiv\dfrac{\pi}{2}$. □.

7. $f(x)=\arcsin\dfrac{2x}{1+x^2}-2\arctan x$.

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\cdot\frac{2(1+x^2)-4x^2}{(1+x^2)^2}-\frac{2}{1+x^2}=\frac{1+x^2}{\sqrt{(1-x^2)^2}}\cdot\frac{2(1-x^2)}{(1+x^2)^2}-\frac{2}{1+x^2}=0.$$

$\Rightarrow$ $f(x)$恒为常数. $f(0)=0\Rightarrow f(x)\equiv 0$. □

8. (1) $\displaystyle\lim_{x\to1}\frac{x-1}{x^n-1}=\lim_{x\to1}\frac{1}{nx^{n-1}}=\frac{1}{n}$.

(2) $\displaystyle\lim_{x\to0}\frac{2^x-1}{3^x-1}=\lim_{x\to0}\frac{2^x\ln2}{3^x\ln3}=\frac{\ln2}{\ln3}$.

(3) $\displaystyle\lim_{x\to1}\frac{\ln x}{x-1}=\lim_{x\to1}\frac{\frac{1}{x}}{1}=1$.

(4) $\displaystyle\lim_{x\to0^+}\frac{\cot x}{\ln x}=\lim_{x\to0^+}\frac{-\frac{1}{\sin^2x}}{\frac{1}{x}}=\lim_{x\to0^+}-\frac{x}{\sin^2x}=-\infty$.

(5) $\displaystyle\lim_{x\to0^+}x^\alpha\ln x=\lim_{x\to0^+}\frac{\ln x}{x^{-\alpha}}=\lim_{x\to0^+}\frac{\frac{1}{x}}{-\alpha x^{-\alpha-1}}=0$.

(6) $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\tan3x}{\tan x}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{3}{\cos^23x}}{\frac{1}{\cos^2x}}=3\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos x}{\cos3x}\right)^2=3\left(\lim_{x\to\frac{\pi}{2}}\frac{-\sin x}{-3\sin3x}\right)^2=\frac{1}{3}$.

(7) $\displaystyle\lim_{x\to0}x^{\sin x}=e^{\lim\limits_{x\to0}\frac{\ln x}{\frac{1}{\sin x}}}=e^{\lim\limits_{x\to0}\frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2x}}}=e^{-\lim\limits_{x\to0}\tan x}=1$.

(8) $\displaystyle\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x=\lim_{t\to0}\frac{a^t-1}{t}=\lim_{t\to0}a^t\ln a=\ln a$.

(9) $\displaystyle\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to0}\frac{e^x-x-1}{x(e^x-1)}=\lim_{x\to0}\frac{e^x-1}{e^x-1+xe^x}=\lim_{x\to0}\frac{e^x}{2e^x+xe^x}=\frac{1}{2}$.

(10) $\lim\limits_{x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x^{100}} = \lim\limits_{t\to+\infty} t^{50}e^{-t} = \lim\limits_{t\to\infty} \dfrac{t^{50}}{e^t} = \lim\limits_{t\to\infty} \dfrac{50!}{e^t} = 0.$

(11) $\lim\limits_{x\to 1} \dfrac{x^x-1}{x\ln x} = \lim\limits_{x\to 1} \dfrac{e^{x\ln x}(\ln x+1)}{\ln x+1} = \lim\limits_{x\to 1} e^{x\ln x} = 1.$

(12) $\lim\limits_{x\to+\infty} \dfrac{x^n}{e^{5x}} = \lim\limits_{x\to+\infty} \dfrac{nx^{n-1}}{5e^{5x}} = \cdots = \lim\limits_{x\to+\infty} \dfrac{n!}{5^n e^{5x}} = 0.$

(13) $\lim\limits_{x\to+\infty} (x+e^x)^{\frac{1}{x}} = \lim\limits_{x\to+\infty} e^{\frac{\ln(x+e^x)}{x}} = e^{\lim\limits_{x\to+\infty} \frac{1+e^x}{x+e^x}} = e^{\lim\limits_{x\to+\infty} \frac{e^x}{1+e^x}} = e.$

(14) $\lim\limits_{x\to 1} \dfrac{\ln\cos(x-1)}{1-\sin(\frac{\pi x}{2})} = \lim\limits_{t\to 0} \dfrac{\ln\cos t}{1-\cos(\frac{\pi}{2}t)} = \lim\limits_{t\to 0} \dfrac{\frac{-\sin t}{\cos t}}{\frac{\pi}{2}\sin(\frac{\pi}{2}t)} = \lim\limits_{t\to 0} -\frac{2}{\pi}\cdot\frac{1}{\cos t}\cdot\frac{t}{\frac{\pi}{2}t} = -\frac{4}{\pi^2}$

9. (1) ~~$\frac{\infty}{\infty}$ 型 可以~~ $\cancel{= \lim\limits_{x\to+\infty} \frac{x-\sin x}{x+\sin x}} = \lim\limits_{x\to\infty} \dfrac{1-\cos x}{1+\cos x} = 2\lim\limits_{x\to\infty} \dfrac{1}{1+\cos x} - 1$ 不存在.

~~如果可以~~ 上下求导,

故不可以用洛必达. $\lim\limits_{x\to\infty} \dfrac{x-\sin x}{x+\sin x} = \lim\limits_{x\to\infty} (1-\frac{\sin x}{x}) \cdot \lim\limits_{x\to\infty} \dfrac{1}{1+\frac{\sin x}{x}} = 1.$

(2) 上下求导, $\lim\limits_{x\to 0} \dfrac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x} = -\lim\limits_{x\to 0} \cos\frac{1}{x}$ 不存在.

故不可以用洛必达. $\lim\limits_{x\to 0} \dfrac{x^2\sin\frac{1}{x}}{\sin x} = \lim\limits_{x\to 0} x\sin\frac{1}{x} = 0.$