高数D第十一次作业解答.

选择题.

4. D.

5. D. $\frac{\partial f}{\partial x} = 2+2x=0 \Rightarrow x=-1$. $\frac{\partial f}{\partial y} = -2-2y=0 \Rightarrow y=-1$.

6. D. $\frac{\partial z}{\partial x} = y = 0$, $\frac{\partial z}{\partial y} = x = 0$. $\forall \varepsilon>0$, $(\frac{\sqrt{\varepsilon}}{2}, \frac{\sqrt{\varepsilon}}{2}) \in U(0,\varepsilon)$, $z = \frac{\varepsilon}{4} > 0$.

$(\frac{\sqrt{\varepsilon}}{2}, -\frac{\sqrt{\varepsilon}}{2}) \in U(0,\varepsilon)$, $z = -\frac{\varepsilon}{4} < 0$.

7. C. $\frac{\partial z}{\partial x} = -2x = 0$, $\frac{\partial z}{\partial y} = -2y = 0 \Rightarrow x=y=0$.

8. C.

9. C. (印刷错误？)

10. B. $\frac{\partial z}{\partial x} = 4x+3y$. $\frac{\partial^2 z}{\partial x \partial y} = 3$.

解答题.

7. (1) $dz = 2xy\,dx + x^2 dy$. (2) $dz = \frac{1}{2\sqrt{xy}} dx - \frac{\sqrt{x}}{2y^{\frac{3}{2}}} dy$.

(3) $dz = -\frac{2y}{(x-y)^2} dx + \frac{2x}{(x-y)^2} dy$. (4) $du = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$.

8. (1) $\frac{dz}{dx} = \frac{u}{\sqrt{u^2+v^2}} \cdot \frac{du}{dx} + \frac{v}{\sqrt{u^2+v^2}} \cdot \frac{dv}{dx} = \frac{\sin x \cos x + e^{2x}}{\sqrt{\sin^2 x + e^{2x}}}$.

(2) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = 2u\ln v \cdot \frac{1}{y} + \frac{u^2}{v} \cdot 3 = \frac{2x\ln(3x-2y)}{y^2} + \frac{3x^2}{y^2(3x-2y)}$.

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = 2u\ln v \cdot (-\frac{x}{y^2}) + \frac{u^2}{v} \cdot (-2) = -\frac{2x^2\ln(3x-2y)}{y^3} - \frac{2x^2}{y^2(3x-2y)}$

(3) $\frac{dz}{dx} = -\frac{v}{u^2} \cdot \frac{du}{dx} + \frac{1}{u}\frac{dv}{dx} = -\frac{1-e^{2x}}{e^x} + \frac{1}{e^x}(-2e^{2x}) = -e^x - e^{-x}$.

(4) $\frac{\partial z}{\partial x} = f'(v) \cdot \frac{\partial v}{\partial x} = -2x f'(y^2-x^2)$. $\frac{\partial z}{\partial y} = 1 + f'(v) \cdot \frac{\partial v}{\partial y} = 1 + 2y f'(y^2-x^2)$.

(5) $\frac{\partial z}{\partial x} = 2uv^3 \cdot \frac{\partial u}{\partial x} + 3u^2v^2 \cdot \frac{\partial v}{\partial x} = 2(x+2y)(x-y)^3 + 3(x+2y)^2(x-y)^2 = (5x+4y)(x+2y)(x-y)^2$.

$\frac{\partial z}{\partial y} = 2uv^3 \cdot \frac{\partial u}{\partial y} + 3u^2v^2 \cdot \frac{\partial v}{\partial y} = 4(x+2y)(x-y)^3 - 3(x+2y)^2(x-y)^2 = (x-10y)(x+2y)(x-y)^2$.

(6) $\frac{dz}{dx} = e^y + x e^y \cdot \varphi'(x) = e^{\varphi(x)}(1 + x\varphi'(x))$.

9. (1) $y + x\frac{dy}{dx} + 2x + 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{y+2x}{x+2y}$.

(2) $y + x\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = \frac{y^2}{1-xy}$.

(3) $\cos y \cdot \frac{dy}{dx} + e^x - y^2 - 2xy\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = \frac{y^2-e^x}{\cos y - 2xy}$.

(4) $\frac{x + y\frac{dy}{dx}}{x^2+y^2} = \frac{\frac{1}{x^2}(x\frac{dy}{dx} - y)}{1+(\frac{y}{x})^2} \Longrightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$.

10. (1) $e^z \cdot \frac{\partial z}{\partial x} - yz - xy\frac{\partial z}{\partial x} = 0 \Longrightarrow \frac{\partial z}{\partial x} = \frac{yz}{e^z - yz}$.

$e^z \cdot \frac{\partial z}{\partial y} - xz - xy\frac{\partial z}{\partial y} = 0 \Longrightarrow \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$.

(2) $3x^2 + 3z^2\frac{\partial z}{\partial x} - 3yz - 3xy\frac{\partial z}{\partial x} = 0 \Longrightarrow \frac{\partial z}{\partial x} = -\frac{x^2-yz}{z^2-xy}$.

$3y^2 + 3z^2\frac{\partial z}{\partial y} - 3xz - 3xy\frac{\partial z}{\partial y} = 0 \Longrightarrow \frac{\partial z}{\partial y} = -\frac{y^2-xz}{z^2-xy}$.

(3) $\cos(x+y-z) \cdot (1 - \frac{\partial z}{\partial x}) = \frac{\partial z}{\partial x} + 1 \Longrightarrow \frac{\partial z}{\partial x} = \frac{\cos(x+y-z)-1}{\cos(x+y-z)+1}$.

$\cos(x+y-z) \cdot (1 - \frac{\partial z}{\partial y}) = \frac{\partial z}{\partial y} \Longrightarrow \frac{\partial z}{\partial y} = \frac{\cos(x+y-z)}{\cos(x+y-z)+1}$.

(4) $-\frac{x}{z^2}\frac{\partial z}{\partial x} + \frac{1}{z} = \frac{1}{z} \cdot \frac{\partial z}{\partial x} \Longrightarrow \frac{\partial z}{\partial x} = \frac{z}{z+x}$.

$-\frac{x}{z^2} \cdot \frac{\partial z}{\partial y} = \frac{1}{z}\frac{\partial z}{\partial y} - \frac{1}{y} \Longrightarrow \frac{\partial z}{\partial y} = \frac{z^2}{y(z+x)}$.

11. $2\cos x\cdot(-\sin x)\,dx+2\cos y\cdot(-\sin y)\,dy+2\cos z\cdot(-\sin z)\,dz=0$

$\Rightarrow dz=-\dfrac{\sin 2x}{\sin 2z}dx-\dfrac{\sin 2y}{\sin 2z}dy.$

12. (1) $\dfrac{\partial z}{\partial x}=2x+y+1=0$, $\dfrac{\partial z}{\partial y}=x+2y-1=0$. $\Rightarrow x=-1, y=1.\ z=0.$

$\dfrac{\partial^2 z}{\partial x^2}=2$, $\dfrac{\partial^2 z}{\partial y^2}=2$, $\dfrac{\partial^2 z}{\partial x\partial y}=1$. $AC-B^2=2\times2-1=3>0$,

故 $(-1,1)$ 为极值点. $A=2>0\Rightarrow z=0$ 为极小值.

(2) $\dfrac{\partial z}{\partial x}=4-2x=0$, $\dfrac{\partial z}{\partial y}=-4-2y=0\Rightarrow x=2, y=-2, z=8.$

$\dfrac{\partial^2 z}{\partial x^2}=-2$, $\dfrac{\partial^2 z}{\partial y^2}=-2$, $\dfrac{\partial^2 z}{\partial x\partial y}=0$. $AC-B^2=4>0$. 故 $(2,-2)$ 为极值点.

$A=-2<0\Rightarrow z=8$ 为极大值.

13. (1) $F(x,y,\lambda)=xy-\lambda(x+y-1)$. $\dfrac{\partial F}{\partial x}=y-\lambda=0$, $\dfrac{\partial F}{\partial y}=x-\lambda=0\Rightarrow x=y=\lambda.$

$x+y=1\Rightarrow x=y=\lambda=\dfrac{1}{2}$, $z=\dfrac{1}{4}$.

(2) $F(x,y,z,\lambda,\mu)=xyz-\lambda(x^2+y^2+z^2-1)-\mu(x+y+z)$.

$\dfrac{\partial F}{\partial x}=yz-2\lambda x-\mu=0$. $\dfrac{\partial F}{\partial y}=zx-2\lambda y-\mu=0$. $\dfrac{\partial F}{\partial z}=xy-2\lambda z-\mu=0$.

$\Rightarrow yz-2\lambda x=zx-2\lambda y=xy-2\lambda z=\mu.$

$\Rightarrow xyz=2\lambda x^2+\mu x=2\lambda y^2+\mu y=2\lambda z^2+\mu z$

$=\dfrac{1}{3}\left(2\lambda(x^2+y^2+z^2)+\mu(x+y+z)\right)=\dfrac{2}{3}\lambda.$ x,y,z 均为 $2\lambda t^2+\mu t-\dfrac{2}{3}\lambda=0$ 的解 ⊛.

~~$2\lambda x^2+\mu x=2\lambda y^2+\mu y\Rightarrow (x-y)(2\lambda(x+y)+\mu)=0$~~.

① 若 $\lambda=0$, 则 $xyz=0$. 不妨 $x=0\Rightarrow y+z=0$, $y^2+z^2=1$. $y=\dfrac{\sqrt2}{2}$, $z=-\dfrac{\sqrt2}{2}$.

② 若 $\lambda\neq0$, 则 ⊛ 有2个~~[illegible]~~根. 若 $x=y=z$ 则由 $x+y+z=0$ 知 $x=y=z=0$, 矛盾!

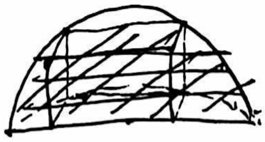
故不妨 x,y 为 ⊛ 的两根, $z=x$. $xy=-\dfrac{1}{3}$ (韦达). $\Rightarrow z=x=-2\lambda$, $y=\dfrac{1}{6\lambda}$.

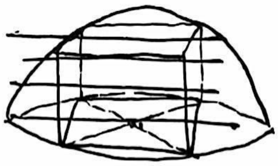
代入得 $-4\lambda+\dfrac{1}{6\lambda}=0$. $\lambda=\pm\dfrac{1}{2\sqrt6}$. $(x,y,z)=\left(-\dfrac{1}{\sqrt6},\dfrac{2}{\sqrt6},-\dfrac{1}{\sqrt6}\right)$ 或 $\left(\dfrac{1}{\sqrt6},-\dfrac{2}{\sqrt6},\dfrac{1}{\sqrt6}\right)$.

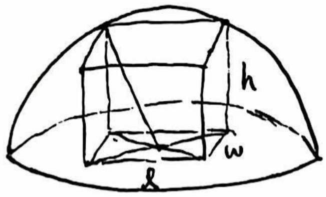
综上，u极大值为 $\frac{2}{3}\times\frac{1}{2\sqrt{6}}=\frac{1}{3\sqrt{6}}$ $\left(x=-\frac{1}{\sqrt{6}},\ y=\frac{2}{\sqrt{6}},\ z=-\frac{1}{\sqrt{6}}\right.$ 及其轮换$)$，u极小值为

$\frac{2}{3}\times\left(-\frac{1}{2\sqrt{6}}\right)=-\frac{1}{3\sqrt{6}}$ $\left(x=\frac{1}{\sqrt{6}},\ y=-\frac{2}{\sqrt{6}},\ z=\frac{1}{\sqrt{6}}\right.$ 及其轮换$)$.

14.







设长宽高为 l, w, h. 由对称性，如图：

$h^2+\left(\frac{w}{2}\right)^2+\left(\frac{l}{2}\right)^2=a^2$. $\quad V=lwh$.

$$F(l,w,h,\lambda)=lwh-\lambda\left(h^2+\left(\frac{w}{2}\right)^2+\left(\frac{l}{2}\right)^2-a^2\right).$$

$\frac{\partial F}{\partial l}=wh-\frac{\lambda}{2}l=0$, $\frac{\partial F}{\partial w}=lh-\frac{\lambda}{2}w=0$, $\frac{\partial F}{\partial h}=lw-2\lambda h=0$.

$\Rightarrow lwh=V=\frac{\lambda}{2}l^2=\frac{\lambda}{2}w^2=2\lambda h^2$. 故 $h=\frac{w}{2}=\frac{l}{2}=\frac{a}{\sqrt{3}}$.

长 $\frac{2}{\sqrt{3}}a$，宽 $\frac{2}{\sqrt{3}}a$，高 $\frac{a}{\sqrt{3}}$ 时 $V=\frac{4}{3\sqrt{3}}a^3$ 最大.

15. 距离 d 即为约束条件 $Ax+By+Cz+D=0$ 下 $\sqrt{x^2+y^2+z^2}$ 的极小值.

$$F(x,y,z,\lambda)=x^2+y^2+z^2-\lambda(Ax+By+Cz+D).$$

$\frac{\partial F}{\partial x}=2x-\lambda A=0$, $\frac{\partial F}{\partial y}=2y-\lambda B=0$, $\frac{\partial F}{\partial z}=2z-\lambda C=0$,

$\Rightarrow x=\frac{\lambda}{2}A$, $y=\frac{\lambda}{2}B$, $z=\frac{\lambda}{2}C$. 代入得 $\lambda=-\frac{2D}{A^2+B^2+C^2}$.

$$d=\sqrt{x^2+y^2+z^2}=\sqrt{\frac{\lambda^2}{4}(A^2+B^2+C^2)}=\frac{|D|}{\sqrt{A^2+B^2+C^2}}.$$

16. 约束条件 $y_2^2=4x_2$, $x_1-y_1+4=0$，找 $(x_1-x_2)^2+(y_1-y_2)^2$ 的极小值.

$$F(x_1,x_2,y_1,y_2,\lambda,\mu)=(x_1-x_2)^2+(y_1-y_2)^2-\lambda(x_1-y_1+4)-\mu(y_2^2-4x_2)$$

$\frac{\partial F}{\partial x_1}=2x_1-2x_2-\lambda=0$, $\frac{\partial F}{\partial x_2}=2x_2-2x_1+4\mu=0$, $\frac{\partial F}{\partial y_1}=2y_1-2y_2+\lambda=0$, $\frac{\partial F}{\partial y_2}=$

$2y_2-2y_1-2\mu y_2=0$. $\Rightarrow y_2=\frac{\lambda}{2\mu}$, $\lambda=4\mu\Rightarrow y_2=2$. $x_2=1$.

$\lambda=2x_1-2=4-2y_1$. $x_1-y_1+4=0$. $x_1=-\frac{1}{2}$, $y_1=\frac{7}{2}$, $\lambda=-3$, $\mu=-\frac{3}{4}$. 所求点 $(1,2)$.

17. $x_1+\cdots+x_n=a$. 则 $\sqrt[n]{x_1\cdots x_n}$ 取最大值 $\Leftrightarrow$ $x_1\cdots x_n$ 取最大值.

$F(x_1,\cdots,x_n,\lambda) = x_1\cdots x_n - \lambda(x_1+\cdots+x_n-a)$.

$\frac{\partial F}{\partial x_i} = \frac{x_1\cdots x_n}{x_i} - \lambda = 0 \Rightarrow x_1\cdots x_n = \lambda x_1 = \cdots = \lambda x_n$.

$\lambda=0 \Rightarrow x_1\cdots x_n=0$. $\lambda\neq 0 \Rightarrow x_1=\cdots=x_n=\frac{a}{n}$, $x_1\cdots x_n=(\frac{a}{n})^n$.

故 $\max\sqrt[n]{x_1\cdots x_n} = \sqrt[n]{(\frac{a}{n})^n} = \frac{a}{n} = \overline{x}$.