高数D第二次作业解答.

17. (1) 不存在. $\lim\limits_{x\to 0^+}\frac{|x|}{x}=1\neq -1=\lim\limits_{x\to 0^-}\frac{|x|}{x}$.

(2) 存在. $\lim\limits_{n\to\infty}\left(1+(-1)^n\frac{1}{n}\right)=1$.

(3) 存在. $\lim\limits_{x\to 1}\operatorname{sgn}^2 x=1$.

(4) 不存在. $\lim\limits_{x\to 3^+}\frac{1}{\sin(x-3)}=+\infty$, $\lim\limits_{x\to 3^-}\frac{1}{\sin(x-3)}=-\infty$.

18. (1) 存在. $\lim\limits_{x\to 0}f(x)=1$.

(2) 不存在. $\lim\limits_{x\to 0^-}f(x)=1\neq 0=\lim\limits_{x\to 0^+}f(x)$.

(3) 不存在. $\lim\limits_{x\to 0^+}\frac{1}{x}=+\infty$, $\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$.

(4) 不存在. $\lim\limits_{x\to 0^+}f(x)=1\neq 0=\lim\limits_{x\to 0^-}f(x)$.

19. (1) $$\lim_{n\to\infty}\frac{4n^2+2}{3n^2+1}=\lim_{n\to\infty}\frac{4+\frac{2}{n^2}}{3+\frac{1}{n^2}}=\frac{4}{3}.$$

(2) $$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0.$$

(3) $$\lim_{x\to 2}\frac{4x+7}{x^2+1}=\frac{4\times 2+7}{2^2+1}=3.$$

(4) $$\lim_{x\to 0}\frac{x^3-4}{4x^2+x-2}=\frac{-4}{-2}=2.$$

(5) $$\lim_{x\to\infty}\frac{x^2-2}{4x^2+x+6}=\lim_{x\to\infty}\frac{1-\frac{2}{x^2}}{4+\frac{1}{x}+\frac{6}{x^2}}=\frac{1}{4}.$$

(6) $$\lim_{x\to\infty}\frac{\sqrt{x+1}-\sqrt{x-1}}{x}=\lim_{x\to\infty}\frac{2}{x\left(\sqrt{x+1}+\sqrt{x-1}\right)}=0.$$

(7) $$\lim_{n\to\infty}\frac{(-2)^n+5^n}{(-2)^{n+1}+5^{n+1}}=\lim_{n\to\infty}\frac{1+\left(-\frac{2}{5}\right)^n}{5+(-2)\cdot\left(-\frac{2}{5}\right)^n}=\frac{1}{5}.$$

(8) $$\lim_{n\to\infty}\frac{\sqrt[3]{n}\sin n}{n+1}=\lim_{n\to\infty}\frac{\sin n}{n^{\frac{2}{3}}+n^{-\frac{1}{3}}}=0.\quad(|\sin n|\le 1)$$

(9) $$\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\quad(x\neq 0)$$

$x=0$: $\lim\limits_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$ 不存在.

(10) $\lim\limits_{n\to\infty} \dfrac{1+2+\cdots+n}{n^2} = \lim\limits_{n\to\infty} \dfrac{n+1}{2n} = \dfrac{1}{2}$.

(11) $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\cdots+\frac{1}{3^n}} = \lim\limits_{n\to\infty} \dfrac{2(1-\frac{1}{2^{n+1}})}{\frac{3}{2}(1-\frac{1}{3^{n+1}})} = \dfrac{4}{3}$.

(12) $\lim\limits_{x\to 1} \dfrac{x^n-1}{x-1} = \lim\limits_{x\to 1} (1+x+\cdots+x^{n-1}) = n$.

(13) $\lim\limits_{x\to -1} \left(\dfrac{1}{x+1} - \dfrac{3}{x^3+1}\right) = \lim\limits_{x\to -1} \dfrac{x^2-x-2}{(x+1)(x^2-x+1)} = \lim\limits_{x\to -1} \dfrac{x-2}{x^2-x+1} = -1$.

(14) $\lim\limits_{x\to 2} \dfrac{x-2}{x+1} = \dfrac{0}{3} = 0$.

(15) $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x^2}-2}{x} = \lim\limits_{x\to 0} \dfrac{x}{\sqrt{4+x^2}+2} = 0$.

(16) $\lim\limits_{x\to\infty} \left(4+\dfrac{1}{x}-\dfrac{1}{x^2}\right) = 4+0-0 = 4$.

证明题：

1. $af(x)+bf(\frac{1}{x}) = \frac{c}{x}$.

令 $x=\frac{1}{t}$ 代入：$af(\frac{1}{t})+bf(t) = ct \Rightarrow af(\frac{1}{x})+bf(x) = cx$.

$\Rightarrow (a^2-b^2)f(x) = a\cdot\frac{c}{x} - b\cdot cx$.

由 $|a|\neq|b|$ 得 $f(x) = \frac{1}{a^2-b^2}\left(\frac{ac}{x}-bcx\right)$.

$f(-x) = \frac{1}{a^2-b^2}\left(-\frac{ac}{x}+bcx\right) = -f(x)$，且 $f(0)=0$. $\square$

2. 否则取 $\varepsilon = \frac{|a|-1}{2} > 0$，$\exists N>0$，$\forall n>N$，

$\left|\frac{x_{n+1}}{x_n}-a\right| < \varepsilon = \frac{|a|-1}{2} \Rightarrow \left|\frac{x_{n+1}}{x_n}\right| \geq |a| - \left|\frac{x_{n+1}}{x_n}-a\right| > \frac{1+|a|}{2} > 1$.

$\Rightarrow \forall n>N$，$|x_n| \geq |x_N| > 0$，与 $\lim\limits_{n\to\infty} x_n = 0$ 矛盾. $\square$

3. 由保号性知 $\exists \delta_0>0$，$\forall x\in U_0(x_0,\delta)$，$f(x)>0$.

$\forall \varepsilon>0$，$\exists \delta_1>0$，$\forall x\in U_0(x_0,\delta_1)$，$|f(x)-A| < \sqrt{A}\varepsilon$.

令 $\delta = \min\{\delta_0,\delta_1\} > 0$，则 $\forall x\in U_0(x_0,\delta)$，$|\sqrt{f(x)}-\sqrt{A}| \leq \dfrac{|f(x)-A|}{|\sqrt{f(x)}+\sqrt{A}|} < \dfrac{\sqrt{A}\varepsilon}{\sqrt{A}} = \varepsilon$.

故 $\lim\limits_{x\to x_0} \sqrt{f(x)} = \sqrt{A}$. $\square$

4. 取 $\varepsilon = \frac{|A|}{2} > 0$，则 $\exists \delta > 0$，$\forall x \in U_0(x_0, \delta)$，有

$$|f(x) - A| < \varepsilon = \frac{|A|}{2} \Rightarrow |f(x)| \geq |A| - |f(x) - A| > \frac{|A|}{2} \Rightarrow \frac{1}{|f(x)|} < \frac{2}{|A|}. \quad \square.$$

5. ① $\forall \varepsilon > 0$，$\exists \delta > 0$，$\forall x \in U_0(x_0, \delta)$，$|g(x)| < \varepsilon \Rightarrow |f(x)| \leq g(x) \leq |g(x)| < \varepsilon$

$$\Rightarrow \lim_{x \to x_0} f(x) = 0.$$

② $-g(x) \leq f(x) \leq g(x)$. $\lim\limits_{x \to x_0} (-g(x)) = 0 = \lim\limits_{x \to x_0} g(x)$. 由夹逼定理知 $\lim\limits_{x \to x_0} f(x) = 0$. $\square$.

6、由保号性知 $\exists X_0 > 0$，$\forall x > X_0$，$|g(x)| \geq \frac{|B|}{2} > 0$.

$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| = \frac{|Bf(x) - Ag(x)|}{|B| \cdot |g(x)|} = \cancel{|(B-A)f(x) + A(f(x)-g(x))|} \; \frac{|B(f(x)-A) + A(B-g(x))|}{|B| \cdot |g(x)|}$$

$$\leq \frac{|f(x) - A|}{|g(x)|} + \frac{|A|}{|B|} \cdot \frac{|B - g(x)|}{|g(x)|}.$$

$\forall \varepsilon > 0$，$\exists X_1 > 0$，$\forall x \geq X_1$，$|f(x) - A| < \frac{|B|}{4} \varepsilon$.

$\forall \varepsilon > 0$，$\exists X_2 > 0$，$\forall x > X_2$，$|g(x) - B| < \frac{|B|^2}{4(|A|+1)} \varepsilon$.

令 $X = \max\{X_0, X_1, X_2\}$，则 $\forall x > X$，有

$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| \leq \frac{\frac{|B|}{4}\varepsilon}{\frac{1}{2}|B|} + \frac{|A|}{|B|} \cdot \frac{1}{\frac{1}{2}|B|} \cdot \frac{|B|^2}{4(|A|+1)} \varepsilon$$

$$= \frac{\varepsilon}{2} + \frac{|A|}{2(|A|+1)} \varepsilon < \varepsilon.$$

故 $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \frac{A}{B}$. $\square$