Kac's Program for the Landau Equation

Xuanrui Feng

Beijing International Center for Mathematical Research
Peking University

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Description of the Physical World

Basic Question: How can we describe our physical world? Or more specifically, how can we describe a complicated system of a vast number of identical interacting particles, such as a rarefied gas or a plasma?

Different scales and governing principles:

Microscopic scale: state vector of all particles $(x_1, v_1, \cdots, x_N, v_N) \in (\mathbb{R}^3 \times \mathbb{R}^3)^N$ governed by Newton's second law of motion. Complete and detailed, but impossible to track and analyze mathematically since $N \sim 10^{23}$.

Macroscopic scale: spatial observables such as the mass $\rho(t, x)$, the velocity $u(t, x)$ and the temperature $T(t, x)$ governed by fluid equations. A simplified continuum and a coarse-grained level. Losing information about the velocity distribution.

Mesoscopic scale: one-particle density function $f(t, x, v)$ governed by kinetic equations. Kinetic description. Intermediate between the above two scales.

Foundings of Kinetic Theory

Fundamental task: formal derivation of governing equations for the one-particle density at the mesoscopic level.

Maxwell (1867): started the systematic study of dynamics of gases; wrote down the first version of the Boltzmann equation in the form of macroscopic observables and discovered the Maxwellian equilibrium.

Boltzmann (1872): wrote down the explicit formula of the Boltzmann equation; established the famous H-theorem and explained the convergence to the Maxwellian equilibrium. Mathematical foundation of second law of thermodynamics.

The Boltzmann equation is useful for a rarefied gas undergoing short-range potentials. But for a plasma with Coulomb potentials, we need a new equation.

Landau (1936): wrote down the Landau equation as an alternative to the Boltzmann equation when the charged particles perform Coulomb collisions in a plasma, by assuming that the grazing collisions prevail.

Hilbert (1900): asked for a rigorous derivation from the microscopic Newton's law to the mesoscopic kinetic equations and then to the macroscopic fluid equations in his sixth problem at the ICM.

Related Works

Microscopic $\rightarrow$ mesoscopic: kinetic limit. Essentially difficult part.

From hard-sphere to Boltzmann: Boltzmann-Grad limit/ low-density limit. Lanford 1975, Illner-Pulvirenti 1989, Gallagher-Saint-Raymond-Texier 2014, Bodineau-Gallagher-Saint-Raymond-Simonella 2023, Deng-Hani-Ma 2024.

From long-range to Landau: weak-coupling limit. Still remains open. A consistency result is given by Bobylev-Pulvirenti-Saffirio 2013 CMP. Asymptotic expansion and Duhamel's principle in the sense of Lanford, but only matches the first-order term.

Mesoscopic $\rightarrow$ macroscopic: hydrodynamic limit. Better understood part.

Bardos-Ukai 1991, Bardos-Golse-Levermore 1991, 1993, Lions-Masmoudi 2001, Golse-Saint-Raymond 2004, Guo-Jang-Jiang 2009, Jiang-Luo-Tang 2024, Deng-Hani-Ma 2025.

Kac's Program

Kac (1956): introduced his seminal program to derive *spatially homogeneous* kinetic equations from many-particle systems.

Kac's model: starting from a artificially designed conservative stochastic model, which simplifies the Newton dynamics to a random Markovian jump process but also preserves some key features: conservation law and deviation angle.

Boltzmann collisional operator: $B(|z|, \cos\theta) = \Gamma(|z|) \cdot b(\cos\theta)$.
Initial state space: N i.i.d. velocities $V_1, \cdots, V_N$ with initial law f_0 in $\mathbb{R}^3$.
Collision rate: for each pair of indices (i, j), choose a random collision time T_{ij} according to the exponential distribution with parameter $\Gamma(|V_i - V_j|)$. Let $T = \min T_{ij}$ be the collision time and change the corresponding pair of velocities.
Post-collisional state space:

$$V_i' = \frac{V_i + V_j}{2} + \frac{|V_i - V_j|}{2}\sigma, \quad V_j' = \frac{V_i + V_j}{2} - \frac{|V_i - V_j|}{2}\sigma,$$

where $\sigma \in \mathbb{S}^2$ is drawn according to the law $b(\cos\theta_{ij})$. θ_{ij} is the deviation angle between the pre-collisional and the post-collisional velocities.

Mean-field Limit and Propagation of Chaos

In Kac's program, the derivation of kinetic equations relies on a *mean-field limit* process, which consists of passing the number of the particles N to infinity but not changing the physical parameters (low-density) or rescaling the interacting kernel (weak-coupling). Derivation of Vlasov equations.

Goal: Assume that initially the empirical measure $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{V_i}$ converges in law to the initial value f_0. Then it also holds at time $t > 0$: $\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{V_i(t)}$ converges in law to f_t, the solution of the Boltzmann equation.

Equivalent statement: Let $F_N(t, v_1, \cdots, v_N)$ be the joint law of the N random variables $(V_1, \cdots, V_N)$ at time t. Assume that initially the k-marginal $F_{N,k}(0)$ converges weakly to $f_0^{\otimes k}$ for any k. Then it also holds at time $t > 0$: $F_{N,k}(t) \rightharpoonup f_t^{\otimes k}$.

This is also known as propagation of (molecular) chaos.

Kac's Program for the Landau Equation

The Landau equation is formally obtained by passing the Boltzmann equation to the *grazing limit*. We can also perform this process at the level of the particle system.

Boltzmann master equation:

$$\partial_t F_N = \frac{1}{N} \sum_{i<j} \int_{\mathbb{S}^2} [F_N(V'_{ij}) - F_N(V)] B(|v_i - v_j|, \cos\theta_{ij}) \, \mathrm{d}\sigma,$$

where we adapt the notation $V = (v_1, \cdots, v_N)$ and $V'_{ij} = (v_1, \cdots, v'_i, \cdots, v'_j, \cdots v_N)$.

$$\begin{array}{ccc} \text{Boltzmann master equation} & \xrightarrow{\text{grazing limit}} & \text{Landau master equation} \\ \Big\downarrow {\scriptstyle \text{mean-field limit}} & & \Big\downarrow {\scriptstyle \text{mean-field limit (goal)}} \\ \text{Boltzmann equation} & \xrightarrow[\text{grazing limit}]{} & \text{Landau equation} \end{array}$$

Landau Equation

Landau Equation: integral-differential form

$$\partial_t f = \nabla_v \cdot \int_{\mathbb{R}^3} a(v-w)\cdot(\nabla_v - \nabla_w) f(v) f(w)\,\mathrm{d}w,$$

or more commonly used divergence form

$$\partial_t f = \nabla \cdot [(a * f)\nabla f - (b * f) f].$$

Here $a(z) = |z|^\gamma(|z|^2 \mathrm{Id} - z \otimes z)$ and $b(z) = \nabla \cdot a(z) = -2|z|^\gamma z$.

The parameter $\gamma \in [-3, 1]$ represents the regularity of the interaction forces.
$\gamma = -3$ corresponds to the most important Coulomb interaction case.
One usually speaks of hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, and very soft potentials if $\gamma \in [-3, -2)$.

Well-posedness: $\gamma = 0$: Villani 1998, $\gamma \in (0, 1]$: Desvillettes-Villani 2000, $\gamma \in [-2, 0)$: Silvestre 2017, $\gamma = [-3, -2)$: Villani 1998, Desvillettes 2015, Guillen-Silvestre 2025.
We take $\gamma = -3$ in this talk.

Landau Master Equation

Landau master equation:

$$\partial_t F_N = \frac{1}{N}\sum_{i<j}^{N} (\nabla_{v_i} - \nabla_{v_j}) \cdot \Big(a(v_i - v_j) \cdot (\nabla_{v_i} - \nabla_{v_j}) F_N \Big).$$

A singular degenerate elliptic linear equation in the divergence form.
We can construct an explicit stochastic particle system associated with the Landau master equation, which conserves the momentum and the kinetic energy almost everywhere.
Kac's particle system:

$$\mathrm{d}V_t^i = \frac{2}{N}\sum_{j=1}^{N} b(V_t^i - V_t^j)\,\mathrm{d}t + \frac{\sqrt{2}}{\sqrt{N}}\sum_{j=1}^{N} \sigma(V_t^i - V_t^j)\,\mathrm{d}Z_t^{i,j}.$$

The vector field σ is given by the square root of the non-negative definite coefficient matrix a. The random factor $Z_t^{i,j}$ is defined in the following anti-symmetric sense: for all pairs of indices $i < j$, $\{Z_t^{i,j}\} = \{B_t^{i,j}\}$ are $N(N-1)/2$ independent 3-dimensional standard Brownian motions, and $Z_t^{j,i} = -Z_t^{i,j}$.

Kac's Program for the Landau Equation

Theorem (F.-Z.Wang (2025))

Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$ be a probability density. Let $F_N \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^{3N}))$ be the unique bounded weak solution to the Landau master equation with initial data $f_0^{\otimes N}$. Let $f \in C^1((0,\infty); \mathcal{S}(\mathbb{R}^3)) \cap L^\infty([0,\infty); L^1 \cap L^\infty(\mathbb{R}^3))$ be the unique bounded smooth solution to the Landau equation with initial data f_0.
Assume that f_0 has finite weighted Fisher information and high-order L^1 moment:

$$\int_{\mathbb{R}^3} \langle v \rangle^3 |\nabla \log f_0|^2 f_0 \,\mathrm{d}v < \infty, \quad \int |v|^m f_0 \,\mathrm{d}v < \infty \text{ for some } m > 6.$$

Then for any $T > 0$, we have propagation of chaos: for any $k \geq 1$, the k-marginal $F_{N,k}$ converges weakly to $f^{\otimes k}$ as $N \to \infty$ on $[0,T] \times \mathbb{R}^{3k}$:

$$\int_{\mathbb{R}^{3k}} (F_{N,k} - f^{\otimes k}) \varphi^{\otimes k} \,\mathrm{d}v_1 \cdots \mathrm{d}v_k \to 0$$

as $N \to \infty$, for any $\varphi \in C_c^\infty(\mathbb{R}^3)$ and any $t \in [0,T]$. Here $\langle v \rangle = \sqrt{1 + |v|^2}$.

Related Works

Kac's program has been developed by McKean 1967, Carlen-Carvalho-Le Roux-Loss-Villani 2010, Mischler-Mouhot 2013, Hauray-Mischler 2014, Carrapatoso 2015.

The study of the Landau master equation: Balescu-Prigogine 1959, Kiessling-Lancelloti 2004 and Miot-Pulvirenti-Saffirio 2011.

The explicit construction of Kac's model for Landau and propagation of chaos: Carrapatoso 2016 for Maxwellian molecules and Fournier-Guillin 2017 for hard potentials.

Particle approximation for Landau: Fontbona-Guérin-Méléard 2009, Fournier 2010, Fournier-Hauray 2016, Carrillo-F.-Guo-Jabin-Wang 2024, Du-Li 2024.

Limitation: all particle approximation results miss the case of very soft potentials. Kac's model has not been justified even for all soft potentials.

Coincidence: Tabary uploaded his preprint on arXiv solving the same problem exactly one day later. A truncation is needed on the particle system.

Duality Formulation

We adapt the duality formulation introduced by Bresch-Duerinckx-Jabin. Consider the backward Kolmogorov equation

$$\partial_t \Phi_N = -\frac{1}{N} \sum_{i<j}^{N} (\nabla_{v_i} - \nabla_{v_j}) \cdot \Big(a(v_i - v_j) \cdot (\nabla_{v_i} - \nabla_{v_j}) \Phi_N \Big),$$

with final data $\Phi_N(T, \cdot) = (C_N^k)^{-1} \sum_{i_1 < \cdots < i_k} \varphi(v_{i_1}) \cdots \varphi(v_{i_k})$.

Motivation: study the evolution of observables instead of the joint law. Φ_N corresponds to the observable along the particle trajectory. This idea is more intrinsic for kinetic equations with various conservation laws.
Such an idea has been considered in Mischler-Mouhot 2013.

By direct computation,

$$\int_{\mathbb{R}^{3k}} F_{N,k}(T)\varphi^{\otimes k} - \int_{\mathbb{R}^{3k}} f(T)^{\otimes k}\varphi^{\otimes k} = \int_{\mathbb{R}^{3N}} F_N(T)\Phi_N(T) - \int_{\mathbb{R}^{3N}} f(T)^{\otimes N}\Phi_N(T)$$

$$= \int_{\mathbb{R}^{3N}} F_N(0)\Phi_N(0) - \int_{\mathbb{R}^{3N}} f(T)^{\otimes N}\Phi_N(T) = -\int_0^T \int_{\mathbb{R}^{3N}} \frac{\mathrm{d}}{\mathrm{d}t} f(t)^{\otimes N}\Phi_N(t).$$

It suffices to prove the asymptotic vanishing of the integral

$$\int_0^T \int_{\mathbb{R}^{3N}} \frac{\mathrm{d}}{\mathrm{d}t} f(t)^{\otimes N}\Phi_N(t).$$

Heuristics: the evolution of a *fixed* observable along the two trajectories asymptotically coincide $\Longrightarrow$ the observable *along the particle trajectory* asymptotically remains unchanged *along the mean-field trajectory*.

By direct computation, this equals (up to a small error)

$$N \int_0^T \int_{\mathbb{R}^{3N}} V_f(v_1, v_2) f^{\otimes N} \Phi_N.$$

Here V_f is a test function gathering the singularity and the unboundedness:

$$\begin{aligned} V_f(v,w) = & -a(v-w) : \Big(\frac{\nabla^2 f}{f}(v) - \nabla \log f(v) \otimes \nabla \log f(w) \Big) \\ & + b(v-w) \cdot (\nabla \log f(v) - \nabla \log f(w)) \\ & + a * f(v) : \frac{\nabla^2 f}{f}(v) - c * f(v). \end{aligned}$$

V_f has a two-side zero-expectation condition.

$$\mathbb{E}_f V_f(\cdot, w) = 0, \quad \mathbb{E}_f V_f(v, \cdot) = 0.$$

Cluster Expansion

We apply the classical cluster expansion to reduce the correlation quantity to the binary correlation.

$$\Phi_N(v_1, \cdots, v_N) = \sum_{n=0}^{N} \sum_{\sigma \in P_n^N} C_{N,n}(v_\sigma)$$

Here P_n^N represents the set of all subsets of $\{1, \cdots, N\}$ with exactly n elements. $v_\sigma = \{v_i : i \in \sigma\}$.
The correlation function $C_{N,n}$ satisfies the zero-expectation condition:

$$\int_{\mathbb{R}^3} C_{N,n}(v_1, \cdots, v_n) f(v_i)\, \mathrm{d}v_i = 0, \forall 1 \le i \le n.$$

Inversion formula:

$$C_{N,n}(v_1, \cdots, v_n) = \sum_{m=0}^{n} (-1)^{n-m} \sum_{\sigma \in P_m^n} \int_{\mathbb{R}^{3(N-m)}} \Phi_N f^{\otimes(N-m)}\, \mathrm{d}v_{\sigma^c}.$$

Applying the cluster expansion formula, we have

$$N \int_0^T \int_{\mathbb{R}^{3N}} V_f(v_1, v_2) f^{\otimes N} \Phi_N = \int_0^T \iint_{\mathbb{R}^6} V_f(v,w) f(v) f(w) C_{N,2}(v,w) \, \mathrm{d}v \, \mathrm{d}w \, \mathrm{d}t.$$

Therefore, it suffices to prove the followin two results:

- $V_f \in L^1([0,T]; L^2(f^{\otimes 2} \, \mathrm{d}v \, \mathrm{d}w))$;
- $NC_{N,2} \rightharpoonup^* 0$ in $L^1([0,T]; L^2(f^{\otimes 2} \, \mathrm{d}v \, \mathrm{d}w))$.

The regularity estimate of V_f requires a new commutator inequality and a second-order Fisher information estimate to deal with the Coulomb singularity.

The weak convergence of $NC_{N,2}$ is obtained by a compactness argument and an energy method for deriving the uniqueness of the limit hierarchy.

Further Discussion

1. Quantitative propagation of chaos for the Landau master equation; Convergence in better sense.
2. Similar results for the Boltzmann equation or the Vlasov-Poisson equation?
3. Central limit theorem/ Gaussian fluctuation/ Large deviation principles for kinetic equations.
4. Derivation of spatially inhomogeneous kinetic equations/ from Newton dynamics under low-density limit or weak-coupling limit.

References

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Thank you!