Quantitative Propagation of Chaos for Singular First-order System on the Whole Space

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First-order System

Consider the large interacting particle system (IPS) with N indistinguishable particles governed by the microscopic SDE system:

$$\mathrm{d}X_i(t) = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_i(t), \quad i = 1, 2, \ldots, N. \quad \text{(IPS)}$$

$X_i(t)$: represents the position of the i-th particle at time t.
$B_i(t)$: N independent standard d-dimensional Brownian motions which model the random collisions on particles.
$\sigma > 0$: represents the viscosity of the dynamics or the inverse temperature.
K: interacting kernel which models the binary interaction forces between particles.
We expect that the particle system converges to its *mean-field limit* as $N \to \infty$, i.e. the McKean-Vlasov system:

$$\mathrm{d}X_t = K * \bar{\rho}_t(X_t)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t, \quad \bar{\rho}_t = \mathrm{Law}(X_t). \quad \text{(MF)}$$

Applying Itô's formula gives the limit Fokker-Planck equation:

$$\partial_t \bar{\rho}_t + \mathrm{div}\Big(\bar{\rho}_t(K * \bar{\rho}_t)\Big) = \sigma\Delta\bar{\rho}_t. \quad \text{(FP)}$$

The Liouville Equation

Statistical Approach: The joint law of N-particle system $\rho_N(t, x_1, \ldots, x_N)$ is governed by the Liouville equation/forward Kolmogorov equation:

$$\partial_t \rho_N + \sum_{i=1}^{N} \mathrm{div}_{x_i}\Big(\rho_N \frac{1}{N}\sum_{j\neq i} K(x_i - x_j)\Big) = \sigma \sum_{i=1}^{N} \Delta_{x_i}\rho_N.$$

The joint law $\rho_N(t,\cdot)$ is symmetric/exchangeable, i.e. $\rho_N(t,\cdot) \in \mathcal{P}_{sym}(\mathbb{R}^{dN})$, since the particles are indistinguishable.
The observables of the particle system (statistical information: temperature, pressure $\cdots$) are contained in the marginals $\rho_{N,k}$ of ρ_N given by

$$\rho_{N,k}(t, x_1, \ldots, x_k) = \int_{\mathbb{R}^{d(N-k)}} \rho_N(t, x_1, \ldots, x_N)\, \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N$$

for any fixed $k = 1, 2, \cdots$
Goal: Establish and quantify the convergence: the k-marginal $\rho_{N,k}(t)$ converges weakly to $\bar{\rho}_t^{\otimes k}$, or the empirical measure $\mu_N^t = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^t}$ converges in law to $\bar{\rho}_t$.

Relative Entropy Method for Propagation of Chaos

Define the relative entropy between ρ_N and $\bar{\rho}^{\otimes N}$ to quantify chaos:

$$\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})(t) \triangleq \int_{\mathbb{R}^{dN}} \rho_N \log \frac{\rho_N}{\bar{\rho}^{\otimes N}} \, \mathrm{d}x_1 \ldots \mathrm{d}x_N.$$

The relative entropy quantity has the sub-additivity property:

$$\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k}) \triangleq \int_{\mathbb{R}^{dN}} \rho_{N,k} \log \frac{\rho_{N,k}}{\bar{\rho}^{\otimes k}} \, \mathrm{d}x_1 \ldots \mathrm{d}x_k \leq \frac{k}{N} \mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N}).$$

Moreover, it controls the square of L^1 distance by the classical Csiszár-Kullback-Pinsker (CKP) inequality

$$\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^1} \leq \sqrt{2\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k})},$$

although it is not itself a distance. One can obtain quantitative strong *propagation of chaos* given the uniform-in-N bound of $\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})$.

This relative entropy method is initiated in the breakthrough paper Jabin-Z.Wang (2018) for first-order systems, but restricted on the torus case.

2D Viscous Vortex Model on the Whole Space $\mathbb{R}^2$

Theorem (F.-Z.Wang (2023))

Assume that ρ_N is an entropy solution to the Liouville equation and that $\bar{\rho}$ solves the limit equation with $\bar{\rho} \geq 0$ and $\int_{\mathbb{R}^2} \bar{\rho}(t,x)\,\mathrm{d}x = 1$. Assume that the initial data $\bar{\rho}_0 \in W^{2,1} \cap W^{2,\infty}(\mathbb{R}^2)$ satisfies the logarithmic growth conditions

$$|\nabla \log \bar{\rho}_0(x)| \lesssim 1 + |x|, \tag{1}$$

$$|\nabla^2 \log \bar{\rho}_0(x)| \lesssim 1 + |x|^2, \tag{2}$$

and the Gaussian upper bound that there exists some $C_0 > 0$ such that

$$\bar{\rho}_0(x) \leq C_0 \exp(-C_0^{-1}|x|^2). \tag{3}$$

Then we have the uniform-in-N bound on the relative entropy

$$\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})(t) \leq M e^{Mt^2}\left(\mathcal{H}_N(\rho_N^0|\bar{\rho}_0^{\otimes N}) + 1\right),$$

where M is some universal constant that only depends on those initial bounds.

2D Viscous Vortex Model with General Circulations

We can further consider the 2D viscous vortex model on the whole space with general circulations for vortices:

$$\mathrm{d}X_i(t) = \frac{1}{N}\sum_{j\neq i}\mathcal{M}_j K(X_i - X_j)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_i(t), \quad i = 1, 2, \ldots, N. \quad \text{(IPS)}$$

Here $\mathcal{M}_i \in \mathbb{R}$ represents the circulation of the i-th vortex, which is assumed to be i.i.d. copies of some compactly supported random variable $\mathcal{M}$ and independent of time t. The different cases of circulations with $\mathcal{M}_i > 0$ and $\mathcal{M}_i < 0$ represent the two different orientations of the point vortices.
The mean-field limit McKean-Vlasov system reads as:

$$\mathrm{d}X_t = K*\omega_t(X_t)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t, \quad \omega_t = \mathbb{E}_{\mathcal{M}}\bar{\rho}_t(\mathcal{M}, X_t), \quad \bar{\rho}_t = \mathrm{Law}(\mathcal{M}, X_t). \quad \text{(MF)}$$

Applying Itô's formula, the vorticity ω_t solves the vorticity formulation of 2D Navier-Stokes equation:

$$\partial_t\omega_t + (K * \omega_t)\cdot\nabla\omega_t = \sigma\Delta\omega_t. \quad \text{(VOR)}$$

Advantage: No longer require $\omega_t \geq 0$ or $\int_{\mathbb{R}^2}\omega_t\,\mathrm{d}x = 1$. More general vorticity.

The Liouville Equation

The joint law of N-particle system $\rho_N(t, z_1, \dots, z_N)$ solves the Liouville equation:

$$\partial_t \rho_N + \frac{1}{N} \sum_{i,j=1}^{N} m_j K(x_i - x_j) \cdot \nabla_{x_i} \rho_N = \sigma \sum_{i=1}^{N} \Delta_{x_i} \rho_N,$$

where $z_i = (m_i, x_i) \in \mathbb{D} = \mathbb{R} \times \mathbb{R}^2$. Now ρ_N is symmetric in z_i but not in x_i.
The k-particle vorticity is given by

$$\omega_{N,k}(t, X^k) = \int_{\mathbb{R}^k \times \mathbb{D}^{(N-k)}} m_1 \cdots m_N \rho_N(t, Z^N) \,\mathrm{d}m_1 \dots \mathrm{d}m_k \,\mathrm{d}z_{k+1} \dots \mathrm{d}z_N$$

for any fixed $k = 1, 2, \dots$. We expect to quantify the convergence from $\omega_{N,k}$ to $\omega_t^{\otimes k}$ in total variation. This is based on the global estimates $\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})$ or local estimates $\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k})$.
Applying Itô's formula, $\bar{\rho}_t$ solves the nonlinear Fokker-Planck equation:

$$\partial_t \bar{\rho}_t + (K * \omega_t) \cdot \nabla_x \bar{\rho}_t = \sigma \Delta_x \bar{\rho}_t. \quad \text{(FP)}$$

Global Relative Entropy Control

Theorem (F.-Z. Wang (2024))

Assume that ρ_N is an entropy solution to the Liouville equation and that $\bar{\rho}$ solves the limit equation with $\bar{\rho} \geq 0$ and $\int_{\mathbb{D}} \bar{\rho}(t,x)\,\mathrm{d}x = 1$. Assume that the initial data $\bar{\rho}_0 \in W^{2,1} \cap W^{2,\infty}(\mathbb{D})$ satisfies the logarithmic growth conditions

$$|\nabla \log \bar{\rho}_0(x)| \lesssim 1 + |x|,$$

$$|\nabla^2 \log \bar{\rho}_0(x)| \lesssim 1 + |x|^2,$$

and the Gaussian upper bound that there exists some $C_0 > 0$ such that

$$\bar{\rho}_0(x) \leq C_0 \exp(-C_0^{-1}|x|^2).$$

Then we have the uniform-in-N bound on the relative entropy

$$\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})(t) \leq M e^{M \log^2(1+t)} \Big(\mathcal{H}_N(\rho_N^0|\bar{\rho}_0^{\otimes N}) + 1 \Big),$$

where M is some universal constant that only depends on those initial bounds.

Sharp Local Estimates and BBGKY Hierarchy

As observed by Lacker (2023), using the sub-additivity property of the relative entropy may lead to suboptimal convergence rate in total variation

$$\|\omega^{N,k} - \omega^{\otimes k}\|_{TV} \lesssim \sqrt{k/N},$$

while the best optimal convergence rate one may expect is actually $O(k/N)$, tested by some simple Gaussian examples.
Regular Kernels $K \in W^{1,\infty}$: Lacker (2023), Lacker-Le Flem (2023).
Singular Kernels: 2D Navier-Stokes with $\mathcal{M}_i \equiv 1$ on $\mathbb{T}^2$ by S. Wang (2024).
The basic idea is to compute the local relative entropy directly using the BBGKY hierarchy solved by the marginals $\rho_{N,k}$.

$$\partial_t \rho_{N,k} + \frac{1}{N}\sum_{i,j=1}^{k} m_j K(x_i - x_j)\cdot \nabla_{x_i}\rho_{N,k} + \frac{N-k}{N}\sum_{i=1}^{k}\int_{\mathbb{D}} m_{k+1} K(x_i - x_{k+1})\cdot \nabla_{x_i}\rho_{N,k+1}\,\mathrm{d}z_{k+1} = \sigma \sum_{i=1}^{k}\Delta_{x_i}\rho_{N,k}.$$

Local Relative Entropy Control

Theorem (F.-Z. Wang (2024))

We further assume that the viscosity constant σ is large enough in the sense of

$$\sigma > \sqrt{2}A\|V\|_\infty,$$

where A is some constant such that $\mathcal{M} \in [-A, A]$. Then we have

$$\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k}) \leq Me^{M\log^2(1+t)}\left(\mathcal{H}_k(\rho_0^{N,k}|\bar{\rho}_0^{\otimes k}) + \frac{k^2}{N^2}\right)$$

for any $t \in [0, T]$, where M is some universal constant depending only on σ, A, C_0 and some Sobolev norms and logarithmic bounds of the initial data $\bar{\rho}_0$.

2D Log Gas on the Whole Space $\mathbb{R}^2$

Theorem (Cai-F.-Gong-Z.Wang (2024))

Assume that ρ_N *is an entropy solution to the Liouville equation and that* $\bar\rho$ *solves the limit equation with* $\bar\rho \geq 0$ *and* $\int_{\mathbb{R}^2} \bar\rho(t,x)\,\mathrm{d}x = 1$. *Assume that the initial data* $\bar\rho_0 \in W^{2,1} \cap W^{2,\infty}(\mathbb{R}^2)$ *satisfies the logarithmic growth conditions*

$$|\nabla \log \bar\rho_0(x)| \lesssim 1 + |x|, \tag{4}$$

$$|\nabla^2 \log \bar\rho_0(x)| \lesssim 1 + |x|^2, \tag{5}$$

and the Gaussian upper bound that there exists some $C_0 > 0$ *such that*

$$\bar\rho_0(x) \leq C_0 \exp(-C_0^{-1}|x|^2). \tag{6}$$

Then for any $\varepsilon > 0$ *we have the uniform-in-N bound on the relative entropy*

$$\mathcal{H}_N(\rho_N|\bar\rho^{\otimes N})(t) \leq M e^{\frac{M}{\varepsilon}t^{\varepsilon}} \Big(\mathcal{E}_N(\rho_N^0|\bar\rho_0^{\otimes N}) + 1\Big),$$

where M *is some universal constant that only depends on those initial bounds.*

Landau Equation with Maxwellian Molecules

Consider the spatially homogeneous Landau equation in dimension d in the following form:

$$\begin{cases} \frac{\partial f}{\partial t} = Q(f,f) = \frac{\partial}{\partial v_\alpha} \int_{\mathbb{R}^d} a_{\alpha\beta}(v - v_*) \Big(f(v_*) \frac{\partial f(v)}{\partial v_\beta} - f(v) \frac{\partial f(v_*)}{\partial v_{*\beta}} \Big) \, \mathrm{d}v_*, \\ f(0,\cdot) = f_0, \end{cases}$$

where $t \geq 0$ and $v \in \mathbb{R}^d$. The coefficient matrix is given by

$$a_{\alpha\beta}(z) = |z|^{\gamma+2} \Pi_{\alpha\beta}(z), \quad \text{with } \Pi_{\alpha\beta}(z) = \delta_{\alpha\beta} - \frac{z_\alpha z_\beta}{|z|^2}. \tag{7}$$

Usually we consider the parameter $\gamma \in [-d, 1]$. The most physically important and meaningful case is when $d = 3$ and $\gamma = -3$ (Landau-Coulombian). Here we consider the case with Maxwellian molecules, i.e. when $\gamma = 0$, for mathematical simplification.

Particle Systems for Landau Equation

We are interested in deriving the Landau equation as the mean-field limit of some many-particle system. Consider the N indistinguishable interacting particle system studied in Fontbona-Guérin-Méléard (2009) and Fournier (2010) that

$$\mathrm{d}V_t^i = \frac{2}{N}\sum_{j=1}^{N} b(V_t^i - V_t^j)\,\mathrm{d}t + \sqrt{2}\Big(\frac{1}{N}\sum_{j=1}^{N} a(V_t^i - V_t^j)\Big)^{\frac{1}{2}}\mathrm{d}B_t^i, \quad i = 1, \ldots, N.$$

We use the convention that $a(0) = 0$ and $b(0) = 0$ to omit the notation $i \neq j$. Applying Itô's formula and the relation $\nabla \cdot a = b$, we can derive the Liouville equation of the N-particle joint distribution $F_N(t, V)$, $V = (v^1, \ldots, v^N)$ on $\mathbb{R}^{dN}$:

$$\partial_t F_N = \sum_{i=1}^{N} \mathrm{div}_{v^i}\Big[\frac{1}{N}\sum_{j=1}^{N} a(v^i - v^j)\nabla_{v^i} F_N - \frac{1}{N}\sum_{j=1}^{N} b(v^i - v^j) F_N\Big],$$

where the initial value is fully factorized as $F_N(0, \cdot) = f_0^{\otimes N}$.

Propagation of Chaos in Relative Entropy

Theorem (Carrillo-F.-Guo-Jabin-Z. Wang (2024))

Assume that F_N is an entropy solution to the Liouville equation and that the classical solution $f \in \mathcal{C}^1([0,T],\mathcal{C}^2(\mathbb{R}^d))$ of the Landau equation satisfies $f \geq 0$ and the conservation laws. Assume that the initial data $f_0 \in \mathcal{C}^2(\mathbb{R}^d)$ satisfies the logarithmic growth conditions

$$|\nabla \log f_0(v)| \lesssim 1 + |v|, \quad |\nabla^2 \log f_0(v)| \lesssim 1 + |v|^2,$$

and the Gaussian upper bound that there exists some $C_0 > 0$ such that

$$f_0(v) \leq C_0 \exp(-C_0^{-1}|v|^2).$$

Then we have the relative entropy estimate

$$\mathcal{H}_N(F_N|f^{\otimes N})(t) \leq \mathcal{H}_N(F_N|f^{\otimes N})(0) + C_T\sqrt{N},$$

where C_T is some constant that depends on those initial bounds and grows polynomially in T.

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Thank you!