Quantitative Propagation of Chaos for Singular First-order System on the Whole Space

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First-order System

Consider the large interacting particle system (IPS) with N indistinguishable particles governed by the microscopic SDE system:

$$\mathrm{d}X_i(t) = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_i(t), \quad i = 1, 2, \ldots, N. \quad \text{(IPS)}$$

$X_i(t)$: represents the position of the i-th particle at time t.
$B_i(t)$: N independent standard d-dimensional Brownian motions which model the random collisions on particles.
$\sigma > 0$: represents the viscosity of the dynamics or the inverse temperature.
K: interacting kernel which models the binary interaction forces between particles.
We expect that the particle system converges to its *mean-field limit* as $N \to \infty$, i.e. the McKean-Vlasov system:

$$\mathrm{d}X_t = K * \bar{\rho}_t(X_t)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t, \quad \bar{\rho}_t = \mathrm{Law}(X_t). \quad \text{(MF)}$$

Applying Itô's formula gives the limit Fokker-Planck equation:

$$\partial_t \bar{\rho}_t + \mathrm{div}\Big(\bar{\rho}_t (K * \bar{\rho}_t)\Big) = \sigma \Delta \bar{\rho}_t. \quad \text{(FP)}$$

The Liouville Equation

Statistical Approach: The joint law of N-particle system $\rho_N(t, x_1, \dots, x_N)$ is governed by the Liouville equation/forward Kolmogorov equation:

$$\partial_t \rho_N + \sum_{i=1}^{N} \mathrm{div}_{x_i}\Big(\rho_N \frac{1}{N}\sum_{j\neq i} K(x_i - x_j)\Big) = \sigma \sum_{i=1}^{N} \Delta_{x_i}\rho_N.$$

The joint law $\rho_N(t,\cdot)$ is symmetric/exchangeable, i.e. $\rho_N(t,\cdot) \in \mathcal{P}_{sym}(\mathbb{R}^{dN})$, since the particles are indistinguishable.
The observables of the particle system (statistical information: temperature, pressure $\cdots$) are contained in the marginals $\rho_{N,k}$ of ρ_N given by

$$\rho_{N,k}(t, x_1, \dots, x_k) = \int_{\mathbb{R}^{d(N-k)}} \rho_N(t, x_1, \dots, x_N)\,\mathrm{d}x_{k+1}\dots\mathrm{d}x_N$$

for any fixed $k = 1, 2, \cdots$
Goal: Establish and quantify the convergence: the k-marginal $\rho_{N,k}(t)$ converges weakly to $\bar\rho_t^{\otimes k}$, or the empirical measure $\mu_N^t = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i^t}$ converges in law to $\bar\rho_t$.

Relative Entropy Method for Propagation of Chaos

Define the global relative entropy (not normalized) between ρ_N and $\bar{\rho}^{\otimes N}$ to quantify chaos:

$$\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})(t) \triangleq \int_{\mathbb{R}^{dN}} \rho_N \log \frac{\rho_N}{\bar{\rho}^{\otimes N}} \, \mathrm{d}x_1 \dots \mathrm{d}x_N.$$

The relative entropy quantity has the sub-additivity property:

$$\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k}) \triangleq \int_{\mathbb{R}^{dk}} \rho_{N,k} \log \frac{\rho_{N,k}}{\bar{\rho}^{\otimes k}} \, \mathrm{d}x_1 \dots \mathrm{d}x_k \leq \frac{k}{N} \mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N}).$$

Moreover, it controls the square of L^1 distance by the classical Csiszár-Kullback-Pinsker (CKP) inequality

$$\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^1} \leq \sqrt{2\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k})},$$

although it is not itself a distance. One can obtain quantitative strong *propagation of chaos* given the uniform-in-N bound of $\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})$.

This relative entropy method is initiated in the breakthrough paper Jabin-Z.Wang (2018) for first-order systems, but restricted to the torus case.

Examples of Models

Regular Kernels $K \in W^{1,\infty}$: classical results by McKean (1967), Dobrushin (1979), Sznitman (1991) via the Coupling Method.

Singular Kernels:

- Conservative flows: 2D Biot-Savart Law $K(x) = \frac{1}{2\pi}\frac{x^{\perp}}{|x|^2}$.

  Osada (1986), Fournier-Hauray-Mischler (2014), Jabin-Z. Wang (2018), Guillin-Le Bris-Monmarché (2024), Shao-Zhao (2024), ...

- Gradient flows: Poisson kernels $K(x) = \pm c_d \frac{x}{|x|^d}$ (repulsive or attractive).
  Riesz kernels $K(x) = \pm c_s \frac{x}{|x|^{s+2}}$ with $0 \leq s < d-2$ (repulsive or attractive).
  Serfaty (2020), Nguyen-Rosenzweig-Serfaty (2022), Bresch-Jabin-Z. Wang (2019a,b,2023), De Courcel-Rosenzweig-Serfaty (2023a,b), ...

Kinetic Models: Landau Equation, Boltzmann Equation.
Fontbona-Guérin-Méléard (2009), Fournier (2010), Mischler-Mouhot (2013), Carrapatoso (2015a,b), Fournier-Hauray (2016), Fournier-Guillin (2017), ...

2D Viscous Vortex Model on the Whole Space $\mathbb{R}^2$

Theorem (F.-Z.Wang (2023))

Assume that ρ_N is an entropy solution to the Liouville equation and that $\bar{\rho}$ solves the limit equation with $\bar{\rho} \geq 0$ and $\int_{\mathbb{R}^2} \bar{\rho}(t,x)\,\mathrm{d}x = 1$. Assume that the initial data $\bar{\rho}_0 \in W^{2,1} \cap W^{2,\infty}(\mathbb{R}^2)$ satisfies the logarithmic growth conditions

$$|\nabla \log \bar{\rho}_0(x)| \lesssim 1 + |x|, \tag{1}$$

$$|\nabla^2 \log \bar{\rho}_0(x)| \lesssim 1 + |x|^2, \tag{2}$$

and the Gaussian upper bound that there exists some $C_0 > 0$ such that

$$\bar{\rho}_0(x) \leq C_0 \exp(-C_0^{-1}|x|^2). \tag{3}$$

Then we have the uniform-in-N bound on the relative entropy

$$\mathcal{H}_N(\rho_N | \bar{\rho}^{\otimes N})(t) \leq M e^{Mt^2} \Big(\mathcal{H}_N(\rho_N^0 | \bar{\rho}_0^{\otimes N}) + 1 \Big),$$

where M is some universal constant that only depends on those initial bounds.

Ideas of the Proof

Compute the time evolution of the relative entropy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})(t) \leq -\sigma\sum_{i=1}^{N}\int_{\mathbb{R}^{2N}}\rho_N\left|\nabla_{x_i}\log\frac{\rho_N}{\bar{\rho}^{\otimes N}}\right|^2\mathrm{d}X^N + \int_{\mathbb{R}^{2N}}\rho_N\Big(\frac{1}{N}\sum_{i,j=1}^{N}\phi(x_i,x_j)\Big)\,\mathrm{d}X^N,$$

where the test function is given by

$$\phi(x,y) = \nabla\log\bar{\rho}(x)\cdot(K*\bar{\rho}(x) - K(x-y)) + (\,\mathrm{div}K*\bar{\rho}(x) - \,\mathrm{div}K(x-y)).$$

Using the symmetrization trick and the divergence-free property, we have

$$\phi(x,y) = -\frac{1}{2}\Big(K(x-y)\cdot(\nabla\log\bar{\rho}(x) - \nabla\log\bar{\rho}(y)) - K*\bar{\rho}(x)\cdot\nabla\log\bar{\rho}(x) - K*\bar{\rho}(y)\cdot\nabla\log\bar{\rho}(y)\Big).$$

Dropping the non-positive relative Fisher information term temporarily (which could be useful to derive uniform-in-time estimates), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N}) \leq \int_{\mathbb{R}^{2N}} \rho_N\Big(\frac{1}{N}\sum_{i,j=1}^{N}\phi(x_i,x_j)\Big)\,\mathrm{d}X^N, \quad (\sim O(N)\ \textit{a priori})$$

where the test function ϕ satisfies the two-side cancellation conditions:

$$\int_{\mathbb{R}^2}\phi(x,y)\bar{\rho}(y)\,\mathrm{d}y = 0, \forall x; \quad \int_{\mathbb{R}^2}\phi(x,y)\bar{\rho}(x)\,\mathrm{d}x = 0, \forall y.$$

We apply the Donsker-Varadhan inequality to change the integral into the expectation with respect to the factorized law $\bar{\rho}^{\otimes N}$, i.e. for any parameter $\eta > 0$,

$$\int_{\mathbb{R}^{2N}}\rho_N\Phi_N\,\mathrm{d}X^N \leq \frac{1}{\eta}\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N}) + \frac{1}{\eta}\log\int_{\mathbb{R}^{2N}}\bar{\rho}^{\otimes N}\exp\left(\eta\Phi_N\right)\,\mathrm{d}X^N,$$

where the test function

$$\Phi_N(x_1,\cdots,x_N) = \frac{1}{N}\sum_{i,j=1}^{N}\phi(x_i,x_j).$$

GOAL: Show the second exponential integral term is uniformly-in-N bounded.

Theorem (Uniform-in-N Large Deviation Type Theorem)

We have

$$\sup_{N\geq 2}\log\int_{\mathbb{R}^{2N}}\bar{\rho}^{\otimes N}\exp\left(\frac{1}{N}\sum_{i,j=1}^{N}\phi(x_i,x_j)\right)\mathrm{d}X^N$$
$$=\sup_{N\geq 2}\log\int_{\mathbb{R}^{2N}}\bar{\rho}^{\otimes N}\exp\left(N\int_{\mathbb{R}^4}\phi(x,y)(\,\mathrm{d}\mu_N-\,\mathrm{d}\bar{\rho})^{\otimes 2}(x,y)\right)\mathrm{d}X^N\leq C\gamma<\infty,$$

provided that ϕ satisfies the two-side cancellation conditions:

$$\int_{\mathbb{R}^2}\phi(x,y)\bar{\rho}(y)\,\mathrm{d}y=0,\forall x,\quad \int_{\mathbb{R}^2}\phi(x,y)\bar{\rho}(x)\,\mathrm{d}x=0,\forall y,$$

and that $\phi(x,\cdot)$ is bounded with $\varphi(x)=\sup_y\phi(x,y)$ satisfying the exponential integrability condition for some $\lambda>0$:

$$\gamma\triangleq\left(\frac{1}{\lambda}\int_{\mathbb{R}^2}e^{\lambda\varphi(x)}\bar{\rho}(x)\,\mathrm{d}x\right)^2<c_0$$

for some fixed small constant $c_0>0$.

The exponential integrability condition is the key ingredient allowing us to migrate the arguments to the whole space, where the test function is no longer bounded. Recall that our test function reads

$$\phi(x,y) = -\frac{1}{2}\Big(K(x-y)\cdot(\nabla\log\bar\rho(x) - \nabla\log\bar\rho(y)) \\ - K*\bar\rho(x)\cdot\nabla\log\bar\rho(x) - K*\bar\rho(y)\cdot\nabla\log\bar\rho(y)\Big).$$

To satisfy our exponential integrability condition, we expect that

- $(K*\bar\rho)\cdot\nabla\log\bar\rho\in L^\infty$;
- $\sup_y |K(x-y)\cdot(\nabla\log\bar\rho(x)-\nabla\log\bar\rho(y))| \le C(1+|x|^2)$;
- $\bar\rho(x)\le Ce^{-C^{-1}|x|^2}$.

Conditions (i)(ii) can be changed into the propagation of the initial conditions to all time $t>0$, say $|\nabla\log\bar\rho(x)|\le C_1(1+|x|)$ and $|\nabla^2\log\bar\rho(x)|\le C_2(1+|x|^2)$, using some parabolic maximum principle.
Condition (iii) follows from the classical Carlen-Loss (1996) arguments.

2D Viscous Vortex Model with General Circulations

We can further consider the 2D viscous vortex model on the whole space with general circulations for vortices:

$$\mathrm{d}X_i(t) = \frac{1}{N}\sum_{j\neq i}\mathcal{M}_j K(X_i - X_j)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_i(t), \quad i = 1, 2, \ldots, N. \quad \text{(IPS)}$$

Here $\mathcal{M}_i \in \mathbb{R}$ represents the circulation of the i-th vortex, which is assumed to be i.i.d. copies of some compactly supported random variable $\mathcal{M}$ and independent of time t. The different cases of circulations with $\mathcal{M}_i > 0$ and $\mathcal{M}_i < 0$ represent the two different orientations of the point vortices.

The mean-field limit McKean-Vlasov system reads as:

$$\mathrm{d}X_t = K*\omega_t(X_t)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t, \quad \omega_t = \mathbb{E}_{\mathcal{M}}\bar{\rho}_t(\mathcal{M}, X_t), \quad \bar{\rho}_t = \mathrm{Law}(\mathcal{M}, X_t). \quad \text{(MF)}$$

Applying Itô's formula, the vorticity ω_t solves the vorticity formulation of 2D Navier-Stokes equation:

$$\partial_t\omega_t + (K * \omega_t)\cdot\nabla\omega_t = \sigma\Delta\omega_t. \quad \text{(VOR)}$$

Advantage: No longer require $\omega_t \geq 0$ or $\int_{\mathbb{R}^2}\omega_t\,\mathrm{d}x = 1$. More general vorticity.

The Liouville Equation

The joint law of N-particle system $\rho_N(t, z_1, \ldots, z_N)$ solves the Liouville equation:

$$\partial_t \rho_N + \frac{1}{N} \sum_{i,j=1}^{N} m_j K(x_i - x_j) \cdot \nabla_{x_i} \rho_N = \sigma \sum_{i=1}^{N} \Delta_{x_i} \rho_N,$$

where $z_i = (m_i, x_i) \in \mathbb{D} = \mathbb{R} \times \mathbb{R}^2$. Now ρ_N is symmetric in z_i but not in x_i.
The k-particle vorticity is given by

$$\omega_{N,k}(t, X^k) = \int_{\mathbb{R}^k \times \mathbb{D}^{(N-k)}} m_1 \cdots m_k \rho_N(t, Z^N) \, \mathrm{d}m_1 \ldots \mathrm{d}m_k \, \mathrm{d}z_{k+1} \ldots \mathrm{d}z_N$$

for any fixed $k = 1, 2, \ldots$. We expect to quantify the convergence from $\omega_{N,k}$ to $\omega_t^{\otimes k}$ in total variation. This is based on the global estimates $\mathcal{H}_N(\rho_N | \bar{\rho}^{\otimes N})$ or local estimates $\mathcal{H}_k(\rho_{N,k} | \bar{\rho}^{\otimes k})$.
Applying Itô's formula, $\bar{\rho}_t$ solves the nonlinear Fokker-Planck equation:

$$\partial_t \bar{\rho}_t + (K * \omega_t) \cdot \nabla_x \bar{\rho}_t = \sigma \Delta_x \bar{\rho}_t. \quad \text{(FP)}$$

Global Relative Entropy Control

Theorem (F.-Z. Wang (2024))

Assume that ρ_N is an entropy solution to the Liouville equation and that $\bar{\rho}$ solves the limit equation with $\bar{\rho} \geq 0$ and $\int_{\mathbb{D}} \bar{\rho}(t,z)\,\mathrm{d}z = 1$. Assume that the initial data $\bar{\rho}_0 \in W_x^{2,1} \cap W_x^{2,\infty}(\mathbb{D})$ satisfies the logarithmic growth conditions

$$|\nabla \log \bar{\rho}_0(m,x)| \lesssim 1 + |x|,$$

$$|\nabla^2 \log \bar{\rho}_0(m,x)| \lesssim 1 + |x|^2,$$

and the Gaussian upper bound that there exists some $C_0 > 0$ such that

$$\bar{\rho}_0(m,x) \leq C_0 \exp(-C_0^{-1}|x|^2).$$

Then we have the uniform-in-N bound on the relative entropy

$$\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})(t) \leq M e^{M \log^2(1+t)} \Big(\mathcal{H}_N(\rho_N^0|\bar{\rho}_0^{\otimes N}) + 1 \Big),$$

where M is some universal constant that only depends on those initial bounds.

Ideas of the Proof

Compute the time evolution of the relative entropy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N})(t) \leq -\sigma \sum_{i=1}^{N} \int_{\mathbb{D}^N} \rho_N \left| \nabla_{x_i} \log \frac{\rho_N}{\bar{\rho}^{\otimes N}} \right|^2 \mathrm{d}Z^N$$

$$-\frac{1}{N}\sum_{i,j=1}^{N} \int_{\mathbb{D}^N} \rho_N \Big(m_j K(x_i - x_j) - K * \omega(x_i) \Big) \cdot \nabla \log \bar{\rho}(z_i)\, \mathrm{d}Z^N.$$

We use integration by parts thanks to $K = \nabla \cdot V$ with $V \in L^\infty$:

$$\frac{1}{N}\sum_{i,j=1}^{N} \int_{\mathbb{D}^N} \Big(m_j V(x_i - x_j) - V * \omega(x_i) \Big) : \Big(\nabla_{x_i} \bar{\rho}^{\otimes N} \otimes \nabla_{x_i} \frac{\rho_N}{\bar{\rho}^{\otimes N}} \Big)\, \mathrm{d}Z^N$$

$$+\frac{1}{N}\sum_{i,j=1}^{N} \int_{\mathbb{D}^N} \rho_N \Big(m_j V(x_i - x_j) - V * \omega(x_i) \Big) : \frac{\nabla^2 \bar{\rho}}{\bar{\rho}}(z_i)\, \mathrm{d}Z^N.$$

The second term is handled similarly by the Large Deviation Type Theorem.

For the first term we use the Cauchy-Schwartz inequality:

$$\frac{\sigma}{4}\sum_{i=1}^{N}\int_{\mathbb{D}^N}\rho_N\left|\nabla_{x_i}\log\frac{\rho_N}{\bar{\rho}^{\otimes N}}\right|^2\mathrm{d}Z^N$$
$$+\frac{2}{\sigma}\sum_{i=1}^{N}\int_{\mathbb{D}^N}\rho_N\left|\frac{1}{N}\sum_{j=1}^{N}\Big(m_jV(x_i-x_j)-V*\omega(x_i)\Big)\right|^2|\nabla\log\bar{\rho}(z_i)|^2\,\mathrm{d}Z^N.$$

Theorem (Uniform-in-N Law of Large Numbers)

Consider any probability density $\bar{\rho}(z)$ on $\mathbb{D}$ and any test function $\phi(z,w)$ satisfying the one-side cancellation condition: $\int_{\mathbb{D}}\phi(z,w)\bar{\rho}(w)\,\mathrm{d}w=0$ for any z, and for the universal constant $c_0>0$ in the Hoeffding's inequality we have

$$\inf\left\{c>0:\int_{\mathbb{D}}\exp\left(\sup_z|\phi(z,w)|^2/c^2\right)\bar{\rho}(w)\,\mathrm{d}w\right\}<\lambda c_0,$$

where $\lambda\in(0,\frac{1}{2})$ is some small constant. Then we have

$$\log\int_{\mathbb{D}^N}\bar{\rho}^{\otimes N}\exp\left(\frac{1}{N}\sum_{j,k=1}^{N}\phi(z_1,z_j)\phi(z_1,z_k)\right)\mathrm{d}Z^N\le\frac{C}{\lambda}<\infty.$$

Logarithmic Estimates and Gaussian Bounds

Proposition 1

Assume that $\bar{\rho}_t(z) \geq 0$ *solves (FP) with* $\int_{\mathbb{D}} \bar{\rho}_t(z)\,\mathrm{d}z = 1$ *for any* t*. Assume that* $\bar{\rho}_0 \in W_x^{2,1} \cap W_x^{2,\infty}(\mathbb{D})$ *satisfies the logarithmic growth conditions*

$$|\nabla \log \bar{\rho}_0(m,x)| \lesssim 1 + |x|, \quad |\nabla^2 \log \bar{\rho}_0(m,x)| \lesssim 1 + |x|^2$$

and the Gaussian upper bound $\bar{\rho}_0(m,x) \leq C_0 \exp(-C_0^{-1}|x|^2)$*, then we have the following logarithmic growth estimates and Gaussian upper bound:*

$$|\nabla \log \bar{\rho}_t(m,x)|^2 \lesssim \frac{1+\log(1+t)}{1+t} + \frac{|x|^2}{(1+t)^2},$$

$$|\nabla^2 \log \bar{\rho}_t(m,x)| \lesssim \frac{1+\log(1+t)}{1+t} + \frac{|x|^2}{(1+t)^2},$$

$$\bar{\rho}_t(m,x) \leq \frac{C}{1+t} \exp\left(-\frac{|x|^2}{8t+C}\right).$$

Ideas of the Proof

Propagating some quantities related to the logarithm along the equation and applying the parabolic maximum principle.

$$(\partial_t - \mathcal{L})\Big(\frac{|\nabla\bar\rho|^2}{\bar\rho}\Big) = -\frac{2\sigma}{\bar\rho}\Big|\partial_{ij}\bar\rho - \frac{\partial_i\bar\rho\partial_j\bar\rho}{\bar\rho}\Big|^2 - \frac{2}{\bar\rho}\nabla\bar\rho\cdot\nabla(K*\omega)\cdot\nabla\bar\rho \le \frac{C}{1+t}\frac{|\nabla\bar\rho|^2}{\bar\rho},$$

$$(\partial_t - \mathcal{L})\Big(\frac{|\nabla^2\bar\rho|^2}{\bar\rho}\Big) \le \frac{C}{1+t}\frac{|\nabla^2\bar\rho|^2}{\bar\rho} + \frac{C}{(1+t)^2}\frac{|\nabla\bar\rho|^2}{\bar\rho},$$

$$(\partial_t - \mathcal{L})(\bar\rho\log\bar\rho) = -\sigma\frac{|\nabla\bar\rho|^2}{\bar\rho}, \quad (\partial_t - \mathcal{L})\Big(\bar\rho(\log\bar\rho)^2\Big) = -\frac{2\sigma}{\bar\rho}|\nabla\bar\rho|^2(1+\log\bar\rho).$$

Constructing the appropriate auxiliary functions to obtain the bounds.

$$F_1(t,x) = \frac{1+t}{C_1}\frac{|\nabla\bar\rho|^2}{\bar\rho} + \bar\rho\log\bar\rho - C_2\bar\rho \le 0.$$

$$F_2(t,x) = \frac{(1+t)^2}{C_3}\frac{|\nabla^2\bar\rho|^2}{\bar\rho} + \frac{1+t}{C_4}\frac{|\nabla\bar\rho|^2}{\bar\rho} - C_5\bar\rho(\log\bar\rho)^2 + C_6\bar\rho\log\bar\rho - C_7\bar\rho \le 0.$$

Sharp Local Estimates and BBGKY Hierarchy

As observed by Lacker (2023), using the sub-additivity property of the relative entropy may lead to suboptimal convergence rate in total variation

$$\|\omega^{N,k} - \omega^{\otimes k}\|_{TV} \lesssim \sqrt{k/N},$$

while the best optimal convergence rate one may expect is actually $O(k/N)$, tested by some simple Gaussian examples.

Regular Kernels $K \in W^{1,\infty}$: Lacker (2023), Lacker-Le Flem (2023).

Singular Kernels: 2D Navier-Stokes with $\mathcal{M}_i \equiv 1$ on $\mathbb{T}^2$ by S. Wang (2024).

The basic idea is to compute the local relative entropy directly using the BBGKY hierarchy solved by the marginals $\rho_{N,k}$.

$$\partial_t \rho_{N,k} + \frac{1}{N} \sum_{i,j=1}^{k} m_j K(x_i - x_j) \cdot \nabla_{x_i} \rho_{N,k} + \frac{N-k}{N} \sum_{i=1}^{k} \int_{\mathbb{D}} m_{k+1} K(x_i - x_{k+1}) \cdot \nabla_{x_i} \rho_{N,k+1} \, \mathrm{d}z_{k+1} = \sigma \sum_{i=1}^{k} \Delta_{x_i} \rho_{N,k}.$$

Local Relative Entropy Control

Theorem (F.-Z. Wang (2024))

We further assume that the viscosity constant σ is large enough in the sense of

$$\sigma > \sqrt{2}A\|V\|_\infty, \ \textit{with } K = \nabla \cdot V,$$

where A is some constant such that $\mathcal{M} \in [-A, A]$. Then we have

$$\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k}) \leq Me^{M\log^2(1+t)}\left(\mathcal{H}_k(\rho^0_{N,k}|\bar{\rho}_0^{\otimes k}) + \frac{k^2}{N^2}\right)$$

for any $t \in [0, T]$, where M is some universal constant depending only on σ, A, C_0 and some Sobolev norms and logarithmic bounds of the initial data $\bar{\rho}_0$.

Ideas of the Proof

Compute the time evolution of the relative entropy

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_k(\rho_{N,k}|\bar{\rho}^{\otimes k}) = -\sigma\sum_{i=1}^{k}\int_{\mathbb{D}^k}\rho_{N,k}\left|\nabla_{x_i}\log\frac{\rho_{N,k}}{\bar{\rho}^{\otimes k}}\right|^2\mathrm{d}Z^k \\
&-\sum_{i=1}^{k}\frac{1}{N}\sum_{j=1}^{k}\int_{\mathbb{D}^k}\rho_{N,k}\Big(m_jK(x_i-x_j)-K*\omega(x_i)\Big)\cdot\nabla\log\bar{\rho}(z_i)\,\mathrm{d}Z^k \\
&-\sum_{i=1}^{k}\frac{N-k}{N}\int_{\mathbb{D}^{k+1}}\rho_{N,k+1}\Big(m_{k+1}K(x_i-x_{k+1})-K*\omega(x_i)\Big)\cdot\nabla\log\bar{\rho}(z_i)\,\mathrm{d}Z^{k+1} \\
&+\sum_{i=1}^{k}\frac{N-k}{N}\int_{\mathbb{D}^{k+1}}\rho_{N,k+1}\Big(m_{k+1}K(x_i-x_{k+1})\cdot\nabla_{x_i}\log\rho_{N,k}\Big)\,\mathrm{d}Z^{k+1}.
\end{aligned}
$$

The second line can be treated similarly as before. We need to rewrite the last two terms into a more compact form.

Using the divergence-free property of K, we group the last two terms by

$$\frac{k(N-k)}{N}\int_{\mathbb{D}^{k+1}}\rho_{N,k}\left(m_{k+1}K(x_1-x_{k+1})\frac{\rho_{N,k+1}}{\rho_{N,k}}-K*\omega(x_1)\right)\cdot\nabla_{x_1}\log\frac{\rho_{N,k}}{\bar{\rho}^{\otimes k}}\,\mathrm{d}Z^{k+1}.$$

By Cauchy-Schwartz inequality, we bound by

$$\begin{aligned}&\frac{\sigma}{4}\sum_{i=1}^{k}\int_{\mathbb{D}^k}\rho_{N,k}\left|\nabla_{x_i}\log\frac{\rho_{N,k}}{\bar{\rho}^{\otimes k}}\right|^2\mathrm{d}Z^k\\ &+\frac{k}{\sigma}\int_{\mathbb{D}^k}\rho_{N,k}\left|\int_{\mathbb{D}}m_{k+1}K(x_1-x_{k+1})\left(\frac{\rho_{N,k+1}}{\rho_{N,k}}-\bar{\rho}(z_{k+1})\right)\mathrm{d}z_{k+1}\right|^2\mathrm{d}Z^k.\end{aligned}$$

Lemma 1

For all $K=\nabla\cdot V$ with $V\in L^\infty(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$ and all regular enough probability density functions m_1,m_2 on $\mathbb{R}^d$, we have for any $\lambda>0$,

$$\begin{aligned}|\langle K,m_1-m_2\rangle|\le{}&\|V\|_\infty\sqrt{\mathcal{I}(m_1|m_2)}\\ &+\lambda^{-1}\sqrt{1+\log\int e^{\lambda^2\|V\|_\infty^2|\nabla\log m_2|^2}\,\mathrm{d}m_2}\sqrt{2\mathcal{H}(m_1|m_2)}.\end{aligned}$$

Thanks to the logarithmic gradient estimate, we conclude by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_k \leq & -\frac{\sigma}{2}\mathcal{I}_k + \frac{A^2\|V\|_\infty^2}{\sigma}\mathcal{I}_{k+1}1_{k<N} + C\frac{1+\log(1+t)}{1+t}\mathcal{H}_k \\ & + C\frac{1+\log(1+t)}{1+t}(\mathcal{H}_{k+1}-\mathcal{H}_k)1_{k<N} + C\frac{k^2}{N^2}\frac{1+\log(1+t)}{1+t}.\end{aligned}$$

The final step is to solve the new ODE hierarchy by using the Grönwall's inequality and carefully computing the iterated integrals.

Proposition 2 (New ODE Hierarchy)

Asuume for $x_k, y_k \geq 0$ and $x_{k+1} \geq x_k$, there exists $c_1 > c_2 \geq 0$ and $h(t) \geq 0$ such that $x_k(t) \leq Ce^{\varphi(t)}\frac{k}{N}$ with $\varphi(t) = C\int_0^t h(s)\,\mathrm{d}s$, and that for all $t \in [0, T]$,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_k \leq -c_1 y_k + c_2 y_{k+1}1_{k<N} + h(t)x_k + kh(t)(x_{k+1}-x_k)1_{k<N} + \frac{k^2}{N^2}h(t), \\ x_k(0) \leq C\frac{k^2}{N^2}. \end{cases}$$

Then there exists some $M > 0$ such that for all $t \in [0, T]$ we have

$$x_k(t) \leq Me^{5\varphi(t)}\frac{k^2}{N^2}.$$

Relative Entropy and Modulated (Potential) Energy

Now we focus on gradient flows, i.e. $K = -\nabla g$.

- Relative Entropy (Jabin-Z.Wang (2018)) :

  The term involving the divergence of K

  $$-\frac{1}{N}\sum_{i\neq j}\int_{\mathbb{R}^{2N}} \rho_N \,\mathrm{div} K(x_i - x_j)\,\mathrm{d}X^N$$

  in the time evolution of the relative entropy is too singular for Coulomb case.

- Modulated Energy (Duerinckx (2016), Serfaty (2020)):

  The modulated (potential) energy is defined as

  $$F(X^N, \bar{\rho}) = \int_{x\neq y} g(x-y)(\,\mathrm{d}\mu_N - \,\mathrm{d}\bar{\rho})^{\otimes 2}(x,y).$$

  Deterministic Coulomb flows or Riesz flows with multiplicative/additive noises can be treated, but only in the sense of weak convergence.

Modulated Free Energy

The combination of relative entropy and the expectation of modulated energy together forms the modulated free energy:

$$\mathcal{E}_N(\rho_N|\bar{\rho}^{\otimes N}) = \frac{1}{N}\mathcal{H}_N(\rho_N|\bar{\rho}^{\otimes N}) + \mathcal{K}_N(\rho_N|\bar{\rho}^{\otimes N}),$$

with

$$\mathcal{K}_N(\rho_N|\bar{\rho}^{\otimes N}) = \frac{1}{\sigma}\mathbb{E}_{\rho_N}(F(X^N,\bar{\rho})) = \frac{1}{\sigma}\mathbb{E}_{\rho_N}\int_{x\neq y} g(x-y)(\,\mathrm{d}\mu_N - \,\mathrm{d}\bar{\rho})^{\otimes 2}(x,y).$$

Under certain Large Deviation Principle, one can control the relative entropy by the modulated free energy and some $O(\frac{1}{N})$ constants. This will recover the strong propagation of chaos.

This modulated free energy method is initiated in the series of paper Bresch-Jabin-Z.Wang (2019a,b,2023) for 2D Log Gas and 2D Patlak-Keller-Segel model, both on the torus.
We migrate the method to the whole space for 2D Log Gas.

2D Log Gas on the Whole Space $\mathbb{R}^2$

Theorem (Cai-F.-Gong-Z.Wang (2024))

Assume that ρ_N is an entropy solution to the Liouville equation and that $\bar\rho$ solves the limit equation with $\bar\rho \geq 0$ and $\int_{\mathbb{R}^2} \bar\rho(t,x)\,\mathrm{d}x = 1$. Assume that the initial data $\bar\rho_0 \in W^{2,1} \cap W^{2,\infty}(\mathbb{R}^2)$ satisfies the logarithmic growth conditions

$$|\nabla \log \bar\rho_0(x)| \lesssim 1 + |x|, \tag{4}$$

$$|\nabla^2 \log \bar\rho_0(x)| \lesssim 1 + |x|^2, \tag{5}$$

and the Gaussian upper bound that there exists some $C_0 > 0$ such that

$$\bar\rho_0(x) \leq C_0 \exp(-C_0^{-1}|x|^2). \tag{6}$$

Then for any $\varepsilon > 0$ we have the uniform-in-N bound on the relative entropy

$$\frac{1}{N}\mathcal{H}_N(\rho_N|\bar\rho^{\otimes N})(t) \leq Me^{\frac{M}{\varepsilon}t^{\varepsilon}}\left(\mathcal{E}_N(\rho_N^0|\bar\rho_0^{\otimes N}) + \frac{1}{N}\right),$$

where M is some universal constant that only depends on those initial bounds.

Main Ideas

We rewrite the modulated free energy as some weighted relative entropy:

$$\mathcal{E}_N(\rho_N|\bar{\rho}^{\otimes N}) = \frac{1}{N}\int_{\mathbb{R}^{2N}} \rho_N \log\left(\frac{\rho_N}{G_{\rho_N}}\frac{G_{\bar{\rho}^{\otimes N}}}{\bar{\rho}^{\otimes N}}\right) \mathrm{d}X^N,$$

where the weight functions are given by

$$G_{\rho_N}(t, X^N) = \exp\Big(-\frac{1}{2N\sigma}\sum_{i\neq j} g(x_i - x_j)\Big),$$

$$G_{\bar{\rho}^{\otimes N}}(t, X^N) = \exp\Big(-\frac{1}{\sigma}\sum_{i=1}^{N} g * \bar{\rho}(x_i) + \frac{N}{2\sigma}\int_{\mathbb{R}^2} g * \bar{\rho}\bar{\rho}\,\mathrm{d}x\Big).$$

The appearance of the weight functions helps cancel the singular divergence term in the gradient system of Coulombian case.

Main Ideas

We compute the time derivative of the modulated free energy by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(\rho_N|\bar{\rho}^{\otimes N})(t) = -\frac{\sigma}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{2N}} \rho_N \left|\nabla_{x_i} \log \frac{\rho_N}{\bar{\rho}^{\otimes N}} - \nabla_{x_i} \log \frac{G_{\rho_N}}{G_{\bar{\rho}^{\otimes N}}}\right|^2 \mathrm{d}X^N$$
$$-\frac{1}{2}\int_{\mathbb{R}^{2N}}\int_{x\neq y} (w(x) - w(y)) \cdot \nabla g(x-y)(\,\mathrm{d}\mu_N - \,\mathrm{d}\bar{\rho})^{\otimes 2}(x,y)\rho_N \,\mathrm{d}X^N,$$

where the vector field is given by $w(x) = \nabla \log \bar{\rho}(x) + \frac{1}{\sigma}\nabla g * \bar{\rho}(x)$.
The second commutator term is in the same form as the previous Large Deviation Type Theorem. The commutator estimates by Rosenzweig-Serfaty (2024) require that $\nabla w \in L^\infty$, which fails on the whole space.
But we can recover the arguments in the 2D vortex model under the same assumptions on the initial data.

Landau Equation with Maxwellian Molecules

Consider the spatially homogeneous Landau equation in dimension d in the following form:

$$\begin{cases} \frac{\partial f}{\partial t} = Q(f,f) = \frac{\partial}{\partial v_\alpha} \int_{\mathbb{R}^d} a_{\alpha\beta}(v-v_*)\left(f(v_*)\frac{\partial f(v)}{\partial v_\beta} - f(v)\frac{\partial f(v_*)}{\partial v_{*\beta}} \right) \mathrm{d}v_*, \\ f(0,\cdot) = f_0, \end{cases}$$

where $t \geq 0$ and $v \in \mathbb{R}^d$. The coefficient matrix is given by

$$a_{\alpha\beta}(z) = |z|^{\gamma+2}\Pi_{\alpha\beta}(z), \quad \text{with } \Pi_{\alpha\beta}(z) = \delta_{\alpha\beta} - \frac{z_\alpha z_\beta}{|z|^2}. \tag{7}$$

Usually we consider the parameter $\gamma \in [-d, 1]$. The most physically important and meaningful case is when $d = 3$ and $\gamma = -3$ (Landau-Coulomb). Here we consider the case with Maxwellian molecules, i.e. when $\gamma = 0$, for mathematical simplification.

Landau Equation with Maxwellian molecules

The Landau system conserves the total mass, momentum and kinetic energy of the gas. We normalize the quantities to

$$\int_{\mathbb{R}^d} f(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}^d} vf(v)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} |v|^2 f(v)\,\mathrm{d}v = d.$$

We can also rewrite the Landau equation in a compact formulation as

$$\partial_t f = \nabla \cdot [(a * f)\nabla f - (b * f)f],$$

or in non-divergence form as

$$\partial_t f = (a * f) : \nabla^2 f - (c * f)f,$$

with the vector field $b = \nabla \cdot a$ and the scalar $c = \nabla \cdot b$, namely

$$b_\alpha(v) = \partial_\beta a_{\alpha\beta}(v) = -(d-1)v_\alpha, \quad c(v) = \partial_\alpha b_\alpha(v) = -d(d-1).$$

Particle Systems for Landau Equation

We are interested in deriving the Landau equation as the mean-field limit of some many-particle system. Consider the N indistinguishable interacting particle system in the sense of Nanbu, studied in Fontbona-Guérin-Méléard (2009) and Fournier (2010), that

$$\mathrm{d}V_t^i = \frac{2}{N}\sum_{j=1}^N b(V_t^i - V_t^j)\,\mathrm{d}t + \sqrt{2}\Big(\frac{1}{N}\sum_{j=1}^N a(V_t^i - V_t^j)\Big)^{\frac{1}{2}}\,\mathrm{d}B_t^i, \quad i = 1, \dots, N.$$

We use the convention that $a(0) = 0$ and $b(0) = 0$ to omit the notation $i \neq j$. Applying Itô's formula and the relation $\nabla \cdot a = b$, we can derive the Liouville equation of the N-particle joint distribution $F_N(t, V)$, $V = (v^1, \dots, v^N)$ on $\mathbb{R}^{dN}$:

$$\partial_t F_N = \sum_{i=1}^N \mathrm{div}_{v^i}\Big[\frac{1}{N}\sum_{j=1}^N a(v^i - v^j)\nabla_{v^i} F_N - \frac{1}{N}\sum_{j=1}^N b(v^i - v^j) F_N\Big],$$

where the initial value is fully factorized as $F_N(0, \cdot) = f_0^{\otimes N}$.

Propagation of Chaos in Relative Entropy

Theorem (Carrillo-F.-Guo-Jabin-Z. Wang (2024))

Assume that F_N is an entropy solution to the Liouville equation and that the classical solution $f \in \mathcal{C}^1([0,T],\mathcal{C}^2(\mathbb{R}^d))$ of the Landau equation satisfies $f \geq 0$ and the conservation laws. Assume that the initial data $f_0 \in \mathcal{C}^2(\mathbb{R}^d)$ satisfies the logarithmic growth conditions

$$|\nabla \log f_0(v)| \lesssim 1 + |v|, \quad |\nabla^2 \log f_0(v)| \lesssim 1 + |v|^2,$$

and the Gaussian upper bound that there exists some $C_0 > 0$ such that

$$f_0(v) \leq C_0 \exp(-C_0^{-1}|v|^2).$$

Then we have the relative entropy estimate

$$\mathcal{H}_N(F_N|f^{\otimes N})(t) \leq \mathcal{H}_N(F_N|f^{\otimes N})(0) + C_T\sqrt{N},$$

where C_T is some constant that depends on those initial bounds and grows polynomially in T.

Main Ideas

We compute the time derivative of the relative entropy by

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N}\mathcal{H}_N(F_N|f^{\otimes N})(t)\\
=&-\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{dN}}F_N\Big(\frac{1}{N}\sum_{j=1}^{N}a(v^i-v^j):\nabla_{v^i}\log\frac{F_N}{f^{\otimes N}}\otimes\nabla_{v^i}\log\frac{F_N}{f^{\otimes N}}\Big)\,\mathrm{d}V\\
&+\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{dN}}F_N\Big[a*f(v^i)-\frac{1}{N}\sum_{j=1}^{N}a(v^i-v^j)\Big]:\frac{\nabla^2 f}{f}(v^i)\,\mathrm{d}V.
\end{aligned}
$$

By Cauchy-Schwartz inequality, the right-hand side can be further bounded by

$$
\Big(\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{dN}}F_N\Big|a*f(v^i)-\frac{1}{N}\sum_{j=1}^{N}a(v^i-v^j)\Big|^2\Big)^{\frac{1}{2}}\Big(\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{dN}}F_N\Big|\frac{\nabla^2 f}{f}(v^i)\Big|^2\Big)^{\frac{1}{2}}.
$$

The first term is vanishing-in-N by a new Law of Large Numbers based on the conservation laws at the particle level, while the second term is uniformly-in-N bounded by estimates on $\nabla\log f$ and $\nabla^2\log f$.

Proposition 3 (Law of Large Numbers)

Assume that $T \sim O(N)$, we have

$$\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{dN}} F_N \Big| a * f(v^i) - \frac{1}{N}\sum_{j=1}^{N} a(v^i - v^j)\Big|^2 \mathrm{d}V \lesssim \frac{1+T}{N}.$$

Using the explicit polynomial form of the test function and carefully computing the evolution of the moments of F_N at all orders.

Proposition 4 (Logarithmic Growth Estimates)

Assume that $f \in \mathcal{C}^1([0,T], \mathcal{C}^2(\mathbb{R}^d))$ with its initial data satisfying the same conditions as in the main result. Then for any $t \in [0,T]$ we have

$$|\nabla \log f| \lesssim 1 + \sqrt{t} + |v|, \quad |\nabla^2 \log f| \lesssim 1 + t + |v|^2.$$

Propagating the same quantities by the usual parabolic maximum principle.

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Thank you!