北京大学高等数学D期末考试参考答案

2022-2023第一学期

本试卷共7道大题，满分100分

一、 求极限（每题5 分，总共20分）

1. $\lim\limits_{x\to 0}\dfrac{(1-\cos x)\ln(1+x^2)}{\sin^2 x(\arctan x)^2}$

$$\begin{aligned}原式&=\lim_{x\to 0}\frac{\frac{1}{2}x^2\cdot x^2}{x^2\cdot x^2}\\&=\frac{1}{2}\end{aligned}$$

2. $\lim\limits_{h\to 0}\displaystyle\int_0^{\pi}\frac{\sin(x+h)-\sin x}{h}\,\mathrm{d}x$

$$\begin{aligned}原式&=\lim_{h\to 0}\frac{1}{h}\left(\int_h^{\pi+h}\sin x\,\mathrm{d}x-\int_0^{\pi}\sin x\,\mathrm{d}x\right)\\&=\lim_{h\to 0}\frac{1}{h}\left(\int_\pi^{\pi+h}\sin x\,\mathrm{d}x-\int_0^{h}\sin x\,\mathrm{d}x\right)\\&=\sin\pi-\sin 0=0\end{aligned}$$

3. $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sqrt{1+x^2+y^2}-1}{|x|+|y|}$

$$\begin{aligned}原式&=\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{(|x|+|y|)(\sqrt{1+x^2+y^2}+1)}\\&=\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{|x|+|y|}\lim_{(x,y)\to(0,0)}\frac{1}{\sqrt{1+x^2+y^2}+1}\\&=0\end{aligned}$$

4. $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)^{x^2y^2}$

取对数后$x^2y^2\ln(x^2+y^2)\le\frac{(x^2+y^2)^2}{4}\ln(x^2+y^2)$. 由于当$(x,y)\to(0,0)$ 时, $x^2+y^2\to 0$, 且$\lim\limits_{t\to 0}\frac{t^2}{4}\ln t=0$, 因此原式$=\mathrm{e}^0=1$.

二、 求积分（每题5 分，总共20分）

1. $\displaystyle\int\frac{x\mathrm{e}^x}{\sqrt{1+\mathrm{e}^x}}\,\mathrm{d}x$

$$
\begin{aligned}
\text{原式} &= 2\int x\,\mathrm{d}\sqrt{1+\mathrm{e}^x} \\
&= 2\left(x\sqrt{1+\mathrm{e}^x} - \int \sqrt{1+\mathrm{e}^x}\,\mathrm{d}x\right) \xlongequal{t=\sqrt{1+\mathrm{e}^x}} 2\left(x\sqrt{1+\mathrm{e}^x} - \int\ t\,\mathrm{d}\ln\left(t^2-1\right)\right) \\
&= 2\left(x\sqrt{1+\mathrm{e}^x} - \int \frac{2t^2}{t^2-1}\,\mathrm{d}t\right) = 2\left(x\sqrt{1+\mathrm{e}^x} - 2t - \ln\left|\frac{t-1}{t+1}\right|\right) + C \\
&= 2\left(x\sqrt{1+\mathrm{e}^x} - 2\sqrt{1+\mathrm{e}^x} - \ln\left|\frac{\sqrt{1+\mathrm{e}^x}-1}{\sqrt{1+\mathrm{e}^x}+1}\right| + C\right)
\end{aligned}
$$

2. $\displaystyle\int \frac{x^4}{(1-x^2)^3}\,\mathrm{d}x$

由分部积分,

$$
\begin{aligned}
\text{原式} &= \frac{1}{4}\int x^3\mathrm{d}\left(\frac{1}{(1-x^2)^2}\right) \\
&= \frac{1}{4}\frac{x^3}{(1-x^2)^2} - \frac{3}{4}\int \frac{x^2}{(1-x^2)^2}\mathrm{d}x \\
&= \frac{1}{4}\frac{x^3}{(1-x^2)^2} - \frac{3}{8}\int x\mathrm{d}\left(\frac{1}{1-x^2}\right) \\
&= \frac{1}{4}\frac{x^3}{(1-x^2)^2} - \frac{3}{8}\frac{x}{1-x^2} + \frac{3}{8}\int \frac{1}{1-x^2}\mathrm{d}x \\
&= \frac{1}{4}\frac{x^3}{(1-x^2)^2} - \frac{3}{8}\frac{x}{1-x^2} + \frac{3}{16}\int \frac{1}{1-x} + \frac{1}{1+x}\mathrm{d}x \\
&= \frac{1}{4}\frac{x^3}{(1-x^2)^2} - \frac{3}{8}\frac{x}{1-x^2} - \frac{3}{16}\ln(1-x) + \frac{3}{16}\ln(1+x) + C
\end{aligned}
$$

3. $\displaystyle\int_{-1}^{1}\left(\frac{\sin^{2023}x}{1+x^2} + \sqrt{1-x^2}\right)\mathrm{d}x$

由于$\dfrac{\sin^{2023}x}{1+x^2}$ 是奇函数, $\sqrt{1-x^2}$ 是偶函数, 故

$$
\int_{-1}^{1}\left(\frac{\sin^{2023}x}{1+x^2} + \sqrt{1-x^2}\right)\,\mathrm{d}x = \int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x = 2\int_{0}^{1}\sqrt{1-x^2}\,\mathrm{d}x
$$

令$x=\sin t$, 三角换元,

$$2\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = 2\int_0^{\pi/2} \cos^2(t)\,\mathrm{d}t = \int_0^{\pi/2} 1+\cos(2t)\,\mathrm{d}t = \frac{\pi}{2}.$$

4. $\displaystyle\int_2^{+\infty} \frac{1}{x^2(1-x)}\mathrm{d}x$

方法1：令$x=\dfrac{1}{t}$, 则$\mathrm{d}x=-\dfrac{1}{t^2}\,\mathrm{d}t$

$$\begin{aligned}
\text{原式} &= \int_{\frac{1}{2}}^{0} \frac{-\frac{1}{t^2}}{\frac{1}{t^2}\left(1-\frac{1}{t}\right)}\mathrm{d}t \\
&= \int_0^{\frac{1}{2}} \frac{t}{t-1}\mathrm{d}t = \int_0^{\frac{1}{2}} \left(1+\frac{1}{t-1}\right)\mathrm{d}t \\
&= t+\ln|t-1|\big|_0^{\frac{1}{2}} = \frac{1}{2}-\ln 2
\end{aligned}$$

方法2：$\dfrac{1}{x^2(1-x)} = \dfrac{1}{x^2}+\dfrac{1}{x}-\dfrac{1}{x-1}$

$$\begin{aligned}
\text{原式} &= \lim_{A\to+\infty}\int_2^A \left(\frac{1}{x^2}+\frac{1}{x}-\frac{1}{x-1}\right)\mathrm{d}x \\
&= \lim_{A\to+\infty}\left(-\frac{1}{x}+\ln\frac{x}{x-1}\right)_2^A \\
&= \frac{1}{2}-\ln 2
\end{aligned}$$

三、 求导数（每题10 分，总共20分）

1. 设方程$2x^2+y^2+z^2-4xz=0$确定了z关于x和y的隐函数$z=f(x,y)$.求$\dfrac{\partial^2 z}{\partial x^2}$

解答：令$F(x,y,z)=2x^2+y^2+z^2-4xz$

则：$F_x' = 4(x-z)$, $F_z' = 2(z-2x)$, $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{2(x-z)}{2x-z}$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{2\left(1-z_x'\right)(2x-z)-2(x-z)\left(2-z_x'\right)}{(2x-z)^2} \\ &= \frac{2\left[x(2z-x)-(x-z)^2\right]}{(2x-z)^3}\end{aligned}$$

2. 已知直角坐标与极坐标之间满足关系$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$，且$z = f(x,y)$是$x$和$y$的函数. 求$z$关于$r$和$\theta$的一阶偏导数和二阶偏导数.

解答: 关于r的一阶偏导数:

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta,\end{aligned} \tag{1}$$

关于r的二阶偏导数:

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta, \tag{2}$$

关于θ的一阶偏导数:

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y} r\cos\theta,\end{aligned} \tag{3}$$

关于θ的二阶偏导数:

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} = & \left[\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial x \partial y} r\cos\theta\right](-r\sin\theta) + \frac{\partial z}{\partial x}(-r\cos\theta) \\ & + \left[\frac{\partial^2 z}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 z}{\partial y^2} r\cos\theta\right] r\cos\theta + \frac{\partial z}{\partial y}(-r\sin\theta) \\ = & \frac{\partial^2 z}{\partial x^2} r^2\sin^2\theta - 2\frac{\partial^2 z}{\partial x \partial y} r^2 \sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2} r^2\cos^2\theta - \frac{\partial z}{\partial x} r\cos\theta - \frac{\partial z}{\partial y} r\sin\theta.\end{aligned} \tag{4}$$

四、 （10 分）求以空间曲面$z = f(x, y) = |xy|$为顶，以区域$D = \{(x, y)|x^2 + y^2 \leq a^2\}$为底的曲顶柱体的体积以及函数$z = f(x, y)$在区域$D$内的平均值.

解答: 记$D_1 = \{(x, y)|x^2 + y^2 \leq a^2, x > 0, y > 0\}$, 则体积

$$\begin{aligned}V = & \iint_D |xy| dxdy \\ = & 4\iint_{D_1} xydxdy \\ = & 4\int_0^a dx \int_0^{\sqrt{a^2-x^2}} xydy \\ = & 4\int_0^a dx \left(\frac{x}{2} y^2\Big|_0^{\sqrt{a^2-x^2}}\right) \\ = & 4\int_0^a \frac{x}{2}(a^2 - x^2)dx \\ = & 2\int_0^a a^2x - x^3 dx \\ = & 2\left(\frac{a^2}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^a \\ = & a^4 - \frac{1}{2}a^4 = \frac{1}{2}a^4.\end{aligned} \tag{5}$$

平均值$= \frac{\frac{1}{2}a^4}{\pi a^2} = \frac{1}{2\pi}a^2$.

五、 (10分)已知长方体的表面积为A，问长（L）宽（W）高（H）分别为多少时体积最大?

解答: 求解条件极值问题

$$\begin{aligned}& \max\ LWH \\ & s.t.\ 2LW + 2LH + 2WH = A.\end{aligned} \tag{6}$$

使用拉格朗日乘数法, 令$F = LWH - \lambda(2LW + 2LH + 2WH - A)$, 有

$$\begin{cases}\frac{\partial F}{\partial L} = WH - 2\lambda(W + H) = 0 \\ \frac{\partial F}{\partial W} = LH - 2\lambda(L + H) = 0 \\ \frac{\partial F}{\partial H} = LW - 2\lambda(L + W) = 0 \\ \frac{\partial F}{\partial \lambda} = -2(LW + LH + WH) + A = 0\end{cases} \tag{7}$$

解得当$L=W=H=\sqrt{\frac{A}{6}}$时体积最大, 此时体积为$V=\left(\frac{A}{6}\right)^{\frac{3}{2}}$.

六、 （10 分）已知二元函数$z=f(x,y)=\begin{cases} \dfrac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

请利用多元函数微分学的知识讨论函数$z=f(x,y)$在$(0,0)$处的连续性和可微性.

解答：令$x=r\cos\theta$, $y=r\sin\theta$

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}} &= \lim_{r\to 0}\frac{r^4\cos^2\theta\sin^2\theta}{r^3} \\
&= \lim_{r\to 0} r\cos^2\theta\sin^2\theta \\
&= 0 = f(0,0)
\end{aligned}
$$

所以，$z=f(x,y)$在$(0,0)$处连续。

若$z=f(x,y)$在$(0,0)$处可微，必有$\Delta z=f'_x(0,0)\Delta x+f'_y(0,0)\Delta y+\o(\rho)$

$$
\begin{aligned}
\lim_{\rho\to 0}\frac{\Delta z-\left[f'_x(0,0)\Delta x+f'_y(0,0)\Delta y\right]}{\rho} &= \lim_{\rho\to 0}\frac{\frac{\Delta x^2\Delta y^2}{\rho^3}}{\rho} \\
&= \lim_{\rho\to 0}\frac{\Delta x^2\Delta y^2}{\rho^4} \\
&\overset{y=kx}{=\!=} \lim_{\rho\to 0}\frac{k^2\Delta x^4}{\Delta x^4(1+k^2)^2} = \frac{k^2}{(1+k^2)^2}
\end{aligned}
$$

极限不存在，所以若$z=f(x,y)$在$(0,0)$处不可微.

七、 （5 分）设f在$[-a,a]$上可积，并为偶函数（即$f(x)=f(-x)$）。证明，

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

解答: 由定积分的性质,

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{-a}^{0} f(x)dx. \tag{8}$$

其中$\int_{-a}^{0} f(x)dx=\int_{a}^{0} -f(-y)dy=\int_{0}^{a} f(-y)dy$, 由于$f$是偶函数, 故$\int_{0}^{a} f(-y)dy=\int_{0}^{a} f(x)dx$, 所以$\int_{-a}^{a} f(x)\,dx=2\int_{0}^{a} f(x)\,dx$.